12th
STD

MATHEMATICS

12th- MATHEMATICS STUDY MATERIAL

KRISHNAGIRI DISTRICT

2023-2024.

CHIEF

Mrs. K.P. MAGESHWARI,
CHIEF EDUCATIONAL OFFICER, KRISHNAGIRI DT

COORDINATORS

Dr. M. VENKATESAN, PA to CEO (HSS) KRISHNAGIRI DT

Mr, S. VADIVELU, APO,CEO OFFICE ,KRISHNAGIRI

Dr. B.J. MURALI, HEADMASTER, G.H.S. SCHOOL, BARUR

Mr.S.JOHN PACKIAM, AHM,MMS,RAJU ST,KRISHNAGIRI

MATHEMATICS SUBJECT COORDINATOR

Dr. B.J. MURALI. HEADMASTER, G.H.S. SCHOOL, BARUR.

SUBJECT TEACHERS

Mr. M. SENCHI, P.G. ASST GBHSS- SANTHUR.

Mr K. RAVIKANNAN P.G ASST. GGHSS – KRISHNAGIRI

Mr. S. VENKATESAN. P.G. ASST, P.R.G.B.H.S. SCHOOL, NAGARASAMPATTI

Mr. N. KALIYAPPAN P.G. ASST, G.H.S. SCHOOL, MORANAHALLI.

Mr. M. ARUNKUMAR P.G. ASST. GHSS ULLUKURUKKAI,

Mr. G SEKAR P.G. ASST GHSS MATHIGIRI.

Mr. S. KATHIRAVAN P.G.ASST. GHSS IKUNDAM

Mr. P RAMESH PG. ASST MUTHAMIZH GHSS-ARASAMPATTI ,

Mr. S .VIJAYAN, P.G.ASST. BARUR

Mr. P.SANGEETHA P.G.ASST. GGHSS KAVERIPATTINAM

12-ஆம் வகுப்பு

ஒரு மதிப்பெண் வினாக்கள்

       12-ஆம் வகுப்பு பாடப்புத்தகத்தில் உள்ள ஒரு மதிப்பெண் வினாக்கள், GeoGebra மென்பொருளின் உதவியோடு,ஒரு வினாவிற்கு சரியான விடையை தேர்வு செய்ய ,அதிகபட்சம் மூன்று வாய்ப்புகள் வழங்கி, மாணவர்களின் கற்றல் ,கற்பித்தல் திறன் அதிகரிக்கும் வகையில் வடிவமைக்கப்பட்டுள்ளது என்பதை தெரிவித்துக்கொள்கிறோம் .

குறிப்பு : Hi-Tech Lab-ல் QR Code -ஐ Scan செய்து அல்லது Link -ஐ click செய்து மாணவர்கள் பயிற்சி செய்யும் விதமாக மென்பொருள் உருவாக்கப்பட்டுள்ளது .

                தமிழ் வழி   ஆங்கில வழி

https://www.geogebra.org/m/svp4anun
https://www.geogebra.org/m/zzajah2u

உருவாக்கம் :

முனைவர்.பொ.ஜெ.முரளி   திரு. நா.காளியப்பன்

தலைமை ஆசிரியர்   முதுகலை ஆசிரியர்
அரசு மேல்நிலைப்பள்ளி, பாரூர்.   அரசு மே.நி.பள்ளி ,
மோரன அள்ளி.

VECTOR ALGEBRA

MATHEMATICS

Important hints:

❖ $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

❖ $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

❖ Work done $W = \vec{F} \cdot \vec{d}$

❖ Torque $\vec{\tau} = \vec{r} \times \vec{F}$

❖ $\vec{a}, \vec{b}$ are perpendicular vectors $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$

❖ $\vec{a}, \vec{b}$ are parallel vectors $\Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

❖ Volume of parallelepiped with coterminous vectors $V = \left|[\vec{a}, \vec{b}, \vec{c}]\right|$

❖ **If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$**

Scalar product (or) dot product
$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

The vector product (or) cross product

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

❖ If θ is the acute angle between two straight lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$, then

$\theta = \cos^{-1}\left(\frac{|\vec{b} \cdot \vec{d}|}{|\vec{b}||\vec{d}|}\right)$

❖ The acute angle θ between the two planes $\vec{r} \cdot \vec{n}_1 = p_1$ and $\vec{r} \cdot \vec{n}_2 = p_2$ is $\theta = \cos^{-1}\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$

❖ If θ is the acute angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = p$, then

$\theta = \sin^{-1}\left(\frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}\right)$

❖ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

$\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Leftrightarrow [\vec{a}, \vec{b}, \vec{c}] = 0 \Leftrightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

❖ **MODEL-I**

Parametric Vector Equation

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$$

Non Parametric Vector Equation

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

Cartesian Equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

❖ **MODEL-II**

Parametric Vector Equation

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$$

Non Parametric Vector Equation

$$(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times \vec{c}\right] = 0$$

Cartesian Equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

❖ **MODEL III**

Parametric Vector Equation

$$\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}$$

Non Parametric Vector Equation

$$(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right] = 0$$

Cartesian Equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

❖ If two lines $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$ and $\frac{x - x_2}{d_1} = \frac{y - y_2}{d_2} = \frac{z - z_2}{d_3}$ intersect, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

MATHEMATICS

5 MARKS

1.By vector method, prove that
$$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$$
Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$
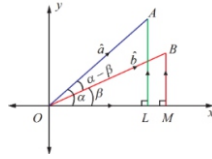
$$\hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$$

$$\hat{b} \cdot \hat{a} = cos(\alpha - \beta) \ - - \rightarrow (1)$$

$$\hat{b} \cdot \hat{a} = (cos\beta\hat{\imath} + sin\beta\hat{\jmath}) \cdot (cos\alpha\hat{\imath} + sin\alpha\hat{\jmath})$$

$$= cos\alpha cos\beta + sin\alpha sin\beta \ - - \rightarrow (2)$$

From (1)&(2)

$$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$$

2.By vector method, prove that
$$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$$
Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$

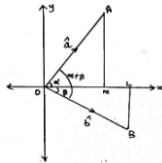$$\hat{b} = cos\beta\hat{\imath} - sin\beta\hat{\jmath}$$

$$\hat{b} \cdot \hat{a} = cos(\alpha + \beta) \ - - \rightarrow (1)$$

$$\hat{b} \cdot \hat{a} = (cos\beta\hat{\imath} - sin\beta\hat{\jmath}) \cdot (cos\alpha\hat{\imath} + sin\alpha\hat{\jmath})$$

$$= cos\alpha cos\beta - sin\alpha sin\beta \ - - \rightarrow (2)$$

From (1)&(2)

$$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$$

3.By vector method, prove that
$$sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$$
Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$

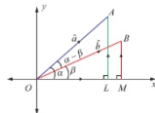$$\hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$$

$$\hat{b} \times \hat{a} = sin(\alpha - \beta)(\hat{k}) \ - - \rightarrow (1)$$

$$\hat{b} \times \hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cos\beta & sin\beta & 0 \\ cos\alpha & sin\alpha & 0 \end{vmatrix}$$

$$= (sin\alpha cos\beta - cos\alpha sin\beta)(\hat{k}) \ - - \rightarrow (2)$$

From (1) & (2)

$$sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$$

4.By vector method, prove that
$$sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$$

Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$
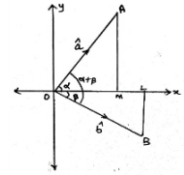
$$\hat{b} = cos\beta\hat{\imath} - sin\beta\hat{\jmath}$$

$$\hat{b} \times \hat{a} = sin(\alpha + \beta)\hat{k} \ - - - \rightarrow (1)$$

$$\hat{b} \times \hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cos\beta & -sin\beta & 0 \\ cos\alpha & sin\alpha & 0 \end{vmatrix}$$

$$= (sin\alpha cos\beta + cos\alpha sin\beta)\hat{k} \ - - - \rightarrow (2)$$

From (1)&(2)

$$sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$$

5.Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.
Soln:

$$\overrightarrow{OA} = \vec{a} \ , \overrightarrow{OB} = \vec{b}, \ \overrightarrow{OC} = \vec{c}$$

AD⊥BC ; BE ⊥CA

To prove CF⊥BA

Case:1 AD⊥BC | Case:2 BE ⊥ CA

$$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0 \qquad\qquad \overrightarrow{OB} \cdot \overrightarrow{CA} = 0$$

$$\overrightarrow{OA} \cdot (\overrightarrow{OC} - \overrightarrow{OB}) = 0 \qquad \overrightarrow{OB} \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = 0$$

$$\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0 \rightarrow (1) \quad \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0 - \rightarrow (2)$$

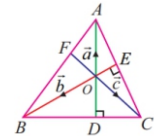From (1) + (2)$\Longrightarrow \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} = 0$

$$(\vec{a} - \vec{b}) \cdot \vec{c} = 0$$

$$(\overrightarrow{OA} - \overrightarrow{OB}) \cdot \overrightarrow{OC} = 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{CF} = 0 \Longrightarrow CF\perp BA$$

Hence, the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

6.If $\vec{a} = \hat{\imath} - \hat{\jmath}, \vec{b} = \hat{\imath} - \hat{\jmath} - 4\hat{k}, \vec{c} = 3\hat{\jmath} - \hat{k}$,and $\vec{d} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$ verify that
$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$$
Soln:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \ - - - - - - \rightarrow (1)$$

$$[\vec{a}, \vec{b}, \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 28 ,$$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 12$$

$$[\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d} = 28(3\hat{j} - \hat{k}) - 12(2\hat{i} + 5\hat{j} + \hat{k})$$

$$= -24\hat{i} + 24\hat{j} - 40\hat{k} - - - - - - - \rightarrow (2)$$

From (1), (2)

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$$

Try Yourself:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$$

7.If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k},$**and**
$\vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$ **verify that**
$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Soln:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = 19\hat{i} - 11\hat{j} - \hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$$= -14\hat{i} - 17\hat{j} - 79\hat{k} \; --> (1)$$

$$\vec{a} \cdot \vec{c} = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (-\hat{i} - 2\hat{j} + 3\hat{k}) = -11$$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (3\hat{i} + 5\hat{j} + 2\hat{k}) = 19$$

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = -11(3\hat{i} + 5\hat{j} + 2\hat{k}) - 19(-\hat{i} - 2\hat{j} + 3\hat{k})$$

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = -14\hat{i} - 17\hat{j} - 79\hat{k} \; ---> (2)$$

From (1),(2)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Try yourself :

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

MODEL- I

8.Find the non-parametric form of Vector Equation, and Cartesian equation of the plane passing through the point (0,1,-5) and parallel to the straight lines
$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + s(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$
$$\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + t(\hat{i} + \hat{j} - \hat{k}).$$

Soln:

$$\vec{a} = 0\hat{i} + \hat{j} - 5\hat{k} \quad \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k} \quad \vec{c} = \hat{i} + \hat{j} - \hat{k}$$

Vector Equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (0\hat{i} + \hat{j} - 5\hat{k}) + s(2\hat{i} + 3\hat{j} + 6\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 0 & y - 1 & z + 5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$(x - 0)(-3 - 6) - (y - 1)(-2 - 6) + (z + 5)(2 - 3) = 0$$

$$-9x + 8y - z - 13 = 0$$

or $\qquad 9x - 8y + z + 13 = 0$

Non Parametirc Vector Equation:

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

$$\vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 13 = 0$$

9.Find the non-parametric form of Vector Equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1} \text{ and } \frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$$

Soln:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k} \quad \vec{b} = 2\hat{i} + 3\hat{j} + \hat{k} \quad \vec{c} = 2\hat{i} - 5\hat{j} - 3\hat{k}$$

Vector Equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (2\hat{i} + 3\hat{j} + 6\hat{k}) + s(2\hat{i} + 3\hat{j} + \hat{k}) + t(2\hat{i} - 5\hat{j} - 3\hat{k})$$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 3 & z - 6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$$

$$(x - 2)(-9 + 5) - (y - 3)(-6 - 2) + (z - 6)(-10 - 6) = 0$$

$$-4x + 8y - 16z + 80 = 0$$

(or) $\qquad x - 2y + 4z - 20 = 0$

Non Parametirc.Vector Equation:

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow \vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) - 20 = 0$$

10. Find the non-parametric form of Vector Equation and Cartesian equation of the plane passing through the point (1,-2,4) and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$.

Soln:

$$\vec{a} = \hat{i} - 2\hat{j} + 4\hat{k} \quad \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} \quad \vec{c} = 3\hat{i} - \hat{j} + \hat{k}$$

Vector Equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(\hat{i} + 2\hat{j} - 3\hat{k}) + t(3\hat{i} - \hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x - 1 & y + 2 & z - 4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$(x - 1)(2 - 3) - (y + 2)(1 + 9) + (z - 4)(-1 - 6) = 0$

$$-x - 10y - 7z + 9 = 0 \qquad (or)$$

$$x + 10y + 7z - 9 = 0$$

Non Parametirc.Vector Equation:

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

$$\vec{r} \cdot (\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$$

11.Find the parametric form of Vector Equation, & Cartesian equations of the plane containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$ **and perpendicular to plane** $\vec{r}.(\hat{i} + 2\hat{j} + \hat{k}) = 8.$

Soln: $\vec{a} = \hat{i} - \hat{j} + 3\hat{k} \quad \vec{b} = 2\hat{i} - \hat{j} + 4k$

$\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$

Vector Equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + s(2\hat{i} - \hat{j} + 3\hat{k}) + t(\hat{i} + 2\hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$(x - 1)(-1 - 8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$

$$-9x + 2y + 5z - 4 = 0$$

$$(or) \quad 9x - 2y - 5z + 4 = 0$$

Non Parametirc.Vector Equation: $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

$$\Rightarrow \vec{r} \cdot (9\hat{i} - 2\hat{j} - 5\hat{k}) + 4 = 0$$

12.Find the non-parametric form of vector eqn, and Cartesian eqns of the plane
$\vec{r} = (6\hat{i} - \hat{j} + \hat{k}) + s(-\hat{i} + 2\hat{j} + \hat{k}) + t(-5\hat{i} - 4\hat{j} - 5\hat{k}).$
Soln:

$\vec{a} = 6\hat{i} - \hat{j} + \hat{k} \quad \vec{b} = -\hat{i} + 2\hat{j} + \hat{k} \quad \vec{c} = -5\hat{i} - 4\hat{j} - 5\hat{k}$

Vector Equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (6\hat{i} - \hat{j} + \hat{k}) + s(-\hat{i} + 2\hat{j} + \hat{k}) + t(-5\hat{i} - 4\hat{j} - 5\hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 6 & y + 1 & z - 1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$(x - 6)(-10 + 4) - (y + 1)(5 + 5) + (z - 1)(4 + 10) = 0$

$$-6x - 10y + 14z + 12 = 0 \quad (or)$$

$$3x + 5y - 7z - 6 = 0$$

Non Parametirc.Vector Equation:

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow \vec{r} \cdot (3\hat{i} + 5\hat{j} - 7\hat{k}) - 6 = 0$$

MODEL-II

13.Find the non-parametric and Cartesian form of the eqn of the plane passing through the points (-1,2,0), (2,2,-1) and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

Soln:

$\vec{a} = -\hat{i} + 2\hat{j} + 0\hat{k} \quad \vec{b} = 2\hat{i} + 2\hat{j} - \hat{k} \quad \vec{c} = \hat{i} + \hat{j} - \hat{k}$

Vector Equation: $\vec{r} = (1 - s)\vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (1 - s)(-\hat{i} + 2\hat{j}) + s(2\hat{i} + 2\hat{j} - \hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x + 1 & y - 2 & z - 0 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$(x + 1)(0 + 1) - (y - 2)(-3 + 1) + (z - 0)(3 - 0) = 0$

$$x + 2y + 3z - 3 = 0$$

Non Parametirc Vector Equation:

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times \vec{c}] = 0$$

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 3$$

14.Find the non-parametric form of vector eqn, Cartesian eqns of the plane passing through the points (2,2,1), (9,3,6) and perpendicular to the plane $2x + 6y + 6z = 9.$
Soln:

$\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k} \quad \vec{b} = 9\hat{i} + 3\hat{j} + b\hat{k} \quad \vec{c} = 2\hat{i} + 6\hat{j} + 6\hat{k}$

Vector Equation: $\vec{r} = (1 - s)\vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (1 - s)(2\hat{i} + 2\hat{j} + \hat{k}) + s(9\hat{i} + 3\hat{j} + 6\hat{k}) + t(2\hat{i} + 6\hat{j} + 6\hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$(x - 2)(6 - 30) - (y - 2)(42 - 10) + (z - 1)(42 - 2) = 0$

$$-24x - 32y + 40z + 72 = 0$$

$$(or) \quad 3x + 4y - 5z - 9 = 0$$

Non Parametirc Vector Equation:

$$(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times \vec{c} \right] = 0$$

$$\vec{r} \cdot \left(3\hat{i} + 4\hat{j} - 5\hat{k} \right) - 9 = 0$$

15. Find parametric form of Vector Equation and Cartesian equations of the plane passing through the points (2, 2,1), (1,-2,3) and parallel to the straight line passing through the points (2, 1, −3) and (−1,5, −8).

Soln:

$$\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k} \quad \vec{b} = \hat{i} - 2\hat{j} + 3\hat{k} \quad \vec{c} = -3\hat{i} + 4\hat{j} - 5\hat{k}$$

Vector Equation: $\vec{r} = (1 - s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1 - s)(2\hat{i} + 2\hat{j} + \hat{k}) + s(\hat{i} - 2\hat{j} + 3\hat{k}) + t(-3\hat{i} + 4\hat{j} - 5\hat{k})$$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x - 2)(20 - 8) - (y - 2)(5 + 6) + (z - 1)(-4 - 12) = 0$$

$$12x - 11y - 16z + 14 = 0$$

Non Parametirc Vector Equation:

$$(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times \vec{c} \right] = 0$$

$$\vec{r} \cdot \left(12\hat{i} - 11\hat{j} - 16\hat{k} \right) + 14 = 0$$

MODEL-III

16. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-Collinear points (3,6, −2), (−1, −2,6) ,and (6, 4, −2).

Soln:

$$\vec{a} = 3\hat{i} + 6\hat{j} - 2\hat{k} \quad \vec{b} = -\hat{i} - 2\hat{j} + 6\hat{k} \quad \vec{c} = 6\hat{i} + 4\hat{j} - 2\hat{k}$$

Vector Equation: $\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1 - s - t)(3\hat{i} + 6\hat{j} - 2\hat{k}) + s(-\hat{i} - 2\hat{j} + 6\hat{k}) + t(6\hat{i} - 4\hat{j} - 2\hat{k})$$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 3 & y - 6 & z + 2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$(x - 3)(0 + 16) - (y - 6)(0 - 24) + (z + 2)(8 + 24) = 0$$

$$16x - 48 + 24y - 144 + 32z + 64 = 0$$

$$(or) \quad 16x + 24y + 32z - 128 = 0$$

$$2x + 3y + 4z - 16 = 0$$

Non Parametirc Vector Equation:

$$(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) \right] = 0$$

$$\vec{r} \cdot \left(2\hat{i} + 3\hat{j} + 8\hat{k} \right) - 16 = 0$$

17. Derive the equation of the plane in the intercept form.

Soln:

$A(a, 0,0), B(0, b, 0), C(0,0, c)$

$\vec{a} = a\hat{i} + 0\hat{j} + 0\hat{k}$,

$\vec{b} = 0\hat{i} + b\hat{j} + 0\hat{k} \ and \ \vec{c} = 0\hat{i} + 0\hat{j} + c\hat{k}$,
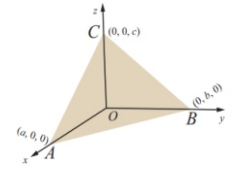
Vector Equation: $\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1 - s - t)a\hat{i} + sb\hat{j} + tc\hat{k}$$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

$$\begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

18. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect and hence find the point of intersection.

Soln:

Condition for intersecting lines

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = s$

$\implies (x, y, z) = (2s + 1, 3s + 2, 4s + 3)$

$\frac{x-4}{5} = \frac{y-1}{2} = z = t$

$\implies (x, y, z) = (5t + 4, 2t + 1, t)$

At the point of intersection

$$(2s + 1, 3s + 2, 4s + 3) = (5t + 4, 2t + 1, t)$$

$$\therefore \text{ we get } s = -1, \ t = -1$$

The point of intersection $(x, y, z) = (-1, -1, -1)$

MATHEMATICS

MATHEMATICS

Try yourself.

Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}, z-1=0$ and $\frac{x-6}{2} = \frac{z-1}{3}, y-2=0$ intersect and hence find the point of intersection. **Hint:** $\frac{x-3}{3} = \frac{y-3}{-1} = \frac{z-1}{0}$ **&** $\frac{x-6}{2} = \frac{z-1}{3} = \frac{y-2}{0}$

Find the parametric form of a vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r} = \hat{i} + 3\hat{j} - \hat{k} + t(2\hat{i} + 3\hat{j} + 2\hat{k})$ and $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$, and perpendicular to both straight lines. **Hint:** $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2}$ **&** $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$

If the two lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-m}{2} = z$ intersect at a point, find the value of m.

ANALYTICAL GEOMETRY

5 Marks

Hints:

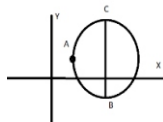**Equation of circle with centre(0,0)and radius r**
$$x^2 + y^2 = r^2$$

Equation of circle with end ponts (x_1, y_1)&(x_2, y_2)
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Ellipse $c^2 = a^2m^2 + b^2$, point of contact $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

Hyperbola $c^2 = a^2m^2 - b^2$, point of contact $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

1.Find the equation of the circle passing through the points $(1,1), (2,-1)$, and $(3,2)$

Soln:

$A(1,1), B(2,-1), C(3,2)$

$M_1 = $ Slope of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{-1-1}{2-1} = -2$

$M_2 = $ Slope of $AC = \frac{2-1}{3-1} = \frac{1}{2}$
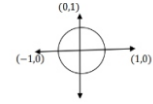
$m_1 \times m_2 = -1 \therefore \angle A = 90°$

End points of diameter B, C,the eqn of circle

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$(x - 2)(x - 3) + (y + 1)(y - 2) = 0$

$x^2 + y^2 - 5x - y + 4 = 0$

2.Find the equation of the circle through the points (1,0),(-1,0), and (0,1).

Soln :

End point of diameter of $(1,0), (-1,0)$

Centre(0,0), radius=1

Equation of circle $x^2 + y^2 = 1$

3.Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$.Also find the coordinates of the point of contact.

Soln :

$x - y + 4 = 0 \qquad\qquad x^2 + 3y^2 = 12$

$\qquad y = x + 4 \qquad\qquad \frac{x^2}{12} + \frac{y^2}{4} = 1$

$m = 1, c = 4 \qquad\qquad a^2 = 12, b^2 = 4$

Condition: $c^2 = a^2m^2 + b^2$

$\qquad c^2 = 16 = a^2m^2 + b^2$

$\qquad x - y + 4 = 0$ is a tangent to $x^2 + 3y^2 = 12$

Point of contact: $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right) = (-3, 1)$

4.Show that the line $5x + 12y = 9$ is a tangent to the hyperbola $x^2 - 9y^2 = 9$,also find point of contact?

Soln :

$5x + 12y = 9 \qquad\qquad x^2 - 9y^2 = 9$

$\Rightarrow y = -\frac{5}{12}x + \frac{3}{4} , \qquad \Rightarrow \frac{x^2}{9} - \frac{y^2}{1} = 9$

$m = -\frac{5}{12}, c = \frac{3}{4} \qquad\qquad a^2 = 9, b^2 = 1$

Condition $c^2 = a^2m^2 - b^2$,

$\Rightarrow c^2 = \frac{9}{16} = a^2m^2 - b^2$

$5x + 12y = 9$ is a tangent to the $x^2 - 9y^2 = 9$

Point of contact is $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right) = (5, -\frac{4}{3})$
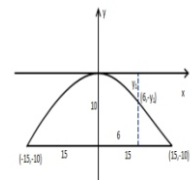
5.A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.

Soln: $x^2 = -4ay ---\rightarrow (1)$

At $(15, -10)$

$(1) \Rightarrow (15)^2 = -4a(-10)$

$\Rightarrow a = \frac{225}{40}$

$(1) \Rightarrow x^2 = -4\left(\frac{225}{40}\right)y - - - \rightarrow (2)$

At $(6, -y_1)$

$$(1) \Rightarrow (6)^2 = -4 \times \frac{225}{40}(-y_1)$$

$$\frac{36 \times 40}{4 \times 225} = y_1 \Rightarrow y_1 = 1.6$$

Required height is $10 - y_1 = 10 - 1.6 = 8.4$ m

6.At a water fountain, water attains a maximum height of $4m$ at horizontal distance of 0.5 m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75 m from the point of origin.

Soln:

$x^2 = -4ay - - - \rightarrow (1)$

At $(-0.5, -4)$

$(1) \Rightarrow (-\frac{1}{2})^2 = -4a(-4)$

$\Rightarrow a = \frac{1}{64}$

$(1) \Rightarrow x^2 = -4\left(\frac{1}{64}\right) - \rightarrow (2)$

At $(0.25, -y_1)$

$(2) \Rightarrow \left(\frac{1}{4}\right)^2 = -4 \times \frac{1}{64}(-y_1) \Rightarrow \frac{64}{4 \times 16} = y_1 \Rightarrow y_1 = 1$

Required distance is $4 - y_1 = 4 - 1 = 3$ m

7.Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every $6m$ along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

Soln:

$x^2 = 4ay - - - - - - - \rightarrow (1)$

At $(30, 13) \Rightarrow 30^2 = 4a(13)$

$\Rightarrow a = \frac{900}{52}$

$(1) \Rightarrow$

$x^2 = 4 \times \frac{900}{52}y \Rightarrow x^2 = \frac{900}{13}y - - - \rightarrow (2)$

(i)At $(6, y_1)$

$(2) \Rightarrow 6^2 = \frac{900}{13}y_1 \Rightarrow \frac{36 \times 13}{900} = y_1 \Rightarrow y_1 = 0.52$
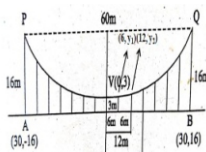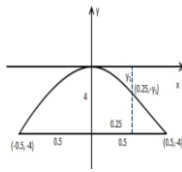
Height of the first cable is $3 + y_1 = 3 + 0.52 = 3.52$

(ii)At $(12, y_2)$

$(2) \Rightarrow 12^2 = \frac{900}{13}y_2 \Rightarrow \frac{144 \times 13}{900} = y_2$

$\Rightarrow y_2 = 2.08$

Height of the second cable is

$3 + y_2 = 3 + 2.08 = 5.08$ m

8.Assume that water issuing from the end of a horizontal pipe, 7.5 m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5 m below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

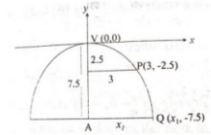**Soln**: $x^2 = -4ay - - - \rightarrow (1)$

At $(3, -2.5)$,

$(1) \Rightarrow (3)^2 = -4a(-2.5)$

$\Rightarrow a = \frac{9}{10}$

$(1) \Rightarrow x^2 = -4\left(\frac{9}{10}\right)y - - \rightarrow (2)$

At $(x_1, -7.5)$ $(2) \Rightarrow (x_1)^2 = -4 \times \frac{9}{10}(-7.5)$

$\Rightarrow (x_1)^2 = 9 \times 3 \Rightarrow x_1 = 3\sqrt{3}$ m

9.On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection

Soln:

$x^2 = -4ay - - - \rightarrow (1)$

At $(6, -4)$

$(1) \Rightarrow (6)^2 = -4a(-4) \Rightarrow a = \frac{36}{16} = \frac{9}{4}$

$(1) \Rightarrow x^2 = -4ay \Rightarrow x^2 = -4\left(\frac{9}{4}\right)y$
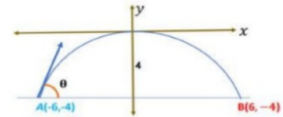
$\Rightarrow x^2 = -9y - - - \rightarrow (2)$

$(2) \; diff. w. r. t. 'x' \Rightarrow 2x = -9\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2x}{-9}$

At $(-6, -4) \Rightarrow \frac{dy}{dx} = \frac{2(-6)}{-9}$

$\frac{dy}{dx} = tan\theta = \frac{4}{3}$

$\therefore \theta = tan^{-1}\left(\frac{4}{3}\right)$

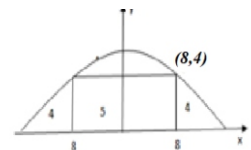**10.A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately. How wide must the opening be?**

Soln:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - - - - \rightarrow (1)$

Given $b = 5$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{5^2} = 1 \text{---} \rightarrow (2)$$

At $(8,4)$

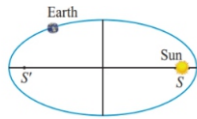$$(2) \Rightarrow \frac{8^2}{a^2} + \frac{4^2}{5^2} = 1$$

$$\frac{8^2}{a^2} = 1 - \frac{16}{25} = \frac{25-16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \frac{8}{a} = \frac{3}{5} \Rightarrow a = \frac{40}{3}$$

Required opening is $2a = \frac{80}{3} = 26.66$ m

11.The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.
Soln.

$SA' = a + c = 152 \times 10^6$

$SA = a - c = 94.5 \times 10^6$

Subtracting $2c = 57.5 \times 10^6 = 575 \times 10^5$ km

Distance from the Sun to the other focus is

$$SS' = 575 \times 10^5 \text{ km.}$$

12.A semielliptical archway over a one-way road has a height of 3 m and a width of 12 m. The truck has a width of 3 m and a height of 2.7 m. Will the truck clear the opening of the archway?
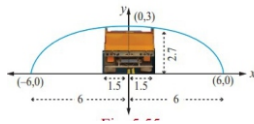**Soln.**
From the diagram $a = 6$ and $b = 3$
Equation of ellipse as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1 \text{---}\rightarrow (1)$

$$At(\tfrac{3}{2}, y_1) \quad (1) \Rightarrow \frac{\left(\frac{3}{2}\right)^2}{36} + \frac{y^2}{9} = 1$$

$$y_1^2 = 9\left(1 - \frac{9}{144}\right)$$

$$y_1^2 = \frac{135}{16}$$

$$y_1 = \frac{\sqrt{135}}{4} = 2.90m$$

The truck will clear the archway.

13.A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of the point P on the rod 0.3 m from the end in contact with x-axis is an ellipse. Find the eccentricity.
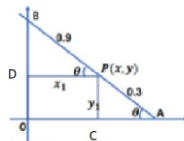**Soln:**

Right angle triangle PAC

$$sin\theta = \frac{y_1}{0.3} \Rightarrow sin^2\theta = \frac{y_1^2}{0.09} \text{---}\rightarrow (1)$$

Right angle triangle BPD

$$cos\theta = \frac{x}{0.9} \Rightarrow cos^2\theta = \frac{x_1^2}{0.81} \text{---}\rightarrow (2)$$

$$(1)^2 + (2)^2 \Rightarrow \frac{x_1^2}{0.81} + \frac{y_1^2}{0.09} = cos^2\theta + sin^2\theta = 1$$

The locus of $(x_1, y_1) is \quad \frac{x^2}{0.81} + \frac{y^2}{0.09} = 1.$

This is ellipse

Eccentricity $e = \sqrt{\frac{a^2 - b^2}{a^2}}$

$$= \sqrt{\frac{0.81 - 0.09}{0.81}} = \sqrt{\frac{0.72}{0.81}}$$

$$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \text{ m}$$

14.Points A and B are $10km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is $6\ km$ closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.
Soln:

$2ae = 10 \Rightarrow ae = 5$;
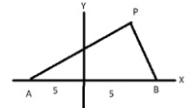
$2a = 6 \Rightarrow a = 3$

$3e = 5 \Rightarrow e = \frac{5}{3} > 1$,

$\therefore$ The curve is an hyperbola.

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(\frac{25}{9} - 1\right)$$

$$\Rightarrow b^2 = 9\left(\frac{25-9}{9}\right) \Rightarrow b^2 = 16$$

Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

15.Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$ The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.
Soln:

Given $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 \rightarrow (1)$

At $(x_1, 50)$

$$(1) \Rightarrow \frac{(x_1)^2}{30^2} - \frac{(50)^2}{44^2} = 1 \Rightarrow \frac{(x_1)^2}{30^2} = 1 + \frac{(50)^2}{44^2}$$
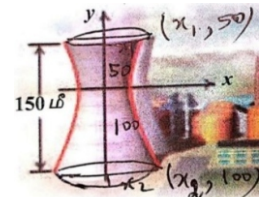
$$x_1 = 45.41m$$

$\therefore$ **the diameter of the top is $2x_1 = 90.82m$**

At $(x_2, 100)$

$$(1) \Rightarrow \frac{(x_2)^2}{30^2} - \frac{(100)^2}{44^2} = 1 \Rightarrow \frac{(x_2)^2}{30^2} = 1 + \frac{(100)^2}{44^2}$$

$$x_2 = 74.49m$$

$\therefore$ **the diameter of the base is $2x_2 = 148.98m$**

COMPLEX NUMBERS

Important Hints:

1. $i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1, i^{4n} = 1$

2. Rectangular form of a complex number is x + iy real part is x, Imaginary part is y.

3. The conjugate of the complex number

z = x + iy is $\bar{z}$= x −iy

4. If $z = x + iy$ then modulus of z is $|z| = \sqrt{x^2 + y^2}$

5. Triangle inequality: For any two complex number z_1 and z_2 , $|z_1 + z_2| \leq |z_1| + |z_2|$ &

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

6. $\sqrt{a \pm ib} = \pm \left[\sqrt{\dfrac{|z| + a}{2}} \pm i \sqrt{\dfrac{|z| - a}{2}} \right]$

7. Additive inverse of **z = −z** , Multiplicative inverse of

z is $\dfrac{1}{z} = \bar{z}$

8. **z is real if any only if z = $\bar{z}$ and**

z is purely imaginary if and only if z = $-\bar{z}$

9. Distance between two complex numbers, z_1 and z_2 is $|z_1 - z_2|$

10. $|z - z_0| = r$ is the complex form of the equation of a circle. Centre is z_0 and radius is r.

5 Marks

1. If $z = x + iy$ is a complex number such that $\left|\dfrac{z-4i}{z+4i}\right| = 1$, S.T. the locus of z is real axis.
Soln:

$z = x + iy$

$\left|\dfrac{z - 4i}{z + 4i}\right| = 1 \Rightarrow |z - 4i| = |z + 4i|$

$|x + iy - 4i| = |x + iy + 4i|$

$|x + i(y - 4)|^2 = |x + i(y + 4)|^2$

$x^2 + (y - 4)^2 = x^2 + (y + 4)^2$

$16y = 0$

$y = 0$

$\therefore z \ is \ real$

2. If $z = x + iy$ is a complex number such that $Im\left(\dfrac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.
Soln:

Given $Im\left(\dfrac{2z+1}{iz+1}\right) = 0$, put $z = x + iy$

$Im\left(\dfrac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$

$Im\left(\dfrac{2x+i2y+1}{ix+i^2y+1}\right) = 0$ $\boxed{Im\left(\dfrac{a+ib}{c+id}\right) = \dfrac{bc-ad}{c^2+d^2}}$

$Im\left(\dfrac{(2x+1)+i2y}{(1-y)+ix}\right) = 0$

$\left(\dfrac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}\right) = 0$

$2y - 2x^2 - 2y^2 - x = 0$ (or)

$2x^2 + 2y^2 + x - 2y = 0$

3. If $z = x + iy$ is a complex number such that $Re\left(\dfrac{z-1}{z+1}\right) = 0$, S.T the locus of z is $x^2 + y^2 = 1$.
Soln:

Given $Re\left(\dfrac{z-1}{z+1}\right) = 0$, put $z = x + iy$

$Re\left(\dfrac{x+iy-1}{x+iy+1}\right) = 0$

$Re\left(\dfrac{(x-1)+iy}{(x+1)+iy}\right) = 0$ $\boxed{Re\left(\dfrac{a+ib}{c+id}\right) = \dfrac{ac+bd}{c^2+d^2}}$

$\left(\dfrac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}\right) = 0$

$x^2 - 1 + y^2 = 0 \implies x^2 + y^2 = 1$

4. If $z = x + iy$ is a complex number such that $arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$, S.T the locus of z is $x^2 + y^2 = 1$.
Soln:

Given $arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$ put $z = x + iy$

$arg\left(\dfrac{x+iy-1}{x+iy+1}\right) = \dfrac{\pi}{2}$

$arg\left(\dfrac{(x-1)+iy}{(x+1)+iy}\right) = \dfrac{\pi}{2}$ $\boxed{arg\left(\dfrac{a+ib}{c+id}\right) = \tan^{-1}\left(\dfrac{bc-ad}{ac+bd}\right)}$

$\tan^{-1}\left(\dfrac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}\right) = \dfrac{\pi}{2}$

$\left(\dfrac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}\right) = \tan\dfrac{\pi}{2} = \infty = \dfrac{1}{0}$

$(x - 1)(x + 1) + y^2 = 0$

$x^2 - 1 + y^2 = 0 \implies x^2 + y^2 = 1$

5. If $z = x + iy$ is a complex number such that $arg\left(\dfrac{z-i}{z+2}\right) = \dfrac{\pi}{4}$, show that the locus of z is $x^2 + y^2 + 3x - 3y + 2 = 0$.
Soln:

Given $arg\left(\dfrac{z-i}{z+2}\right) = \dfrac{\pi}{4}$ put $z = x + iy$

$arg\left(\dfrac{x+iy-i}{x+iy+2}\right) = \dfrac{\pi}{4}$

$arg\left(\dfrac{x+i(y-1)}{(x+2)+iy}\right) = \dfrac{\pi}{4}$ $\boxed{arg\left(\dfrac{a+ib}{c+id}\right) = \tan^{-1}\left(\dfrac{bc-ad}{ac+bd}\right)}$

$\tan^{-1}\left(\dfrac{(x+2)(y-1)-xy}{x(x+2)+y(y-1)}\right) = \dfrac{\pi}{4}$

MATHEMATICS

$\left(\frac{(x+2)(y-1)-xy}{x(x+2)+y(y-1)}\right) = tan\frac{\pi}{4} = 1$

$(x+2)(y-1) - xy = x(x+2) + y(y-1)$

$x^2 + y^2 + 3x - 3y + 2 = 0$

Try yourself

If $z = x + iy$ is a complex number such that $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, show that the locus of z is $\sqrt{3}x^2 + \sqrt{3}y^2 - 2y - 3 = 0$.

6.If $z = 3 + 2i$, represent the complex numbers z, iz, and $z + iz$ in one Argand plane. S.t. these complex numbers form the vertices of an isosceles right triangle.
Soln:

Given, $z = 3 + 2i$

Then $iz = i(3 + 2i) = 3i - 2 = -2 + 3i$

$z + iz = 1 + 5i$

Let $z_1 = z = 3 + 2i$, $z_2 = iz = -2 + 3i$,

$z_3 = z + iz = 1 + 5i$

$AB = |z_1 - z_2| = |(3+2i)-(-2+3i)|$

$= |5 - i| = \sqrt{(5)^2 + (-1)^2} = \sqrt{26}$

$BC = |z_2 - z_3| = |(-2+3i) - (1+5i)|$

$= |-3 - 2i| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

$CA = |z_3 - z_1| = |(1+5i) - (3+2i)|$

$= |-2 + 3i| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

$BC^2 + CA^2 = AB^2 = 26$

∴ Given complex numbers form the vertices of an isosceles right triangle.

7.Show that the points $1, \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.
Soln:

Let $z_1 = 1$ $z_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ $z_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$AB = |z_1 - z_2| = |1 - (\frac{-1}{2} + i\frac{\sqrt{3}}{2})| = \sqrt{3}$

$BC = |z_2 - z_3| = |(\frac{-1}{2} + i\frac{\sqrt{3}}{2}) - (\frac{1}{2} - i\frac{\sqrt{3}}{2})| = |0 + i\sqrt{3}| = \sqrt{3}$

$CA = |z_3 - z_1| = |(\frac{-1}{2} - i\frac{\sqrt{3}}{2}) - 1| = |\frac{-3}{2} - i\frac{\sqrt{3}}{2}| = \sqrt{3}$

AB=BC=CA ∴ Given points are the vertices of an equilateral triangle.

8.If z_1, z_2 and z_3 are three complex numbers S.T $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, show that $|9z_1z_2 + 4z_1z_3 + z_yz_3| = 6$.
Soln:

Given $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$

∵ $|z|^2 = z\bar{z}$, $z_1\bar{z_1} = 1$, $z_2\bar{z_2} = 4$, $z_3\bar{z_3} = 9$

$z_1 = \frac{1}{\bar{z_1}}, z_2 = \frac{4}{\bar{z_2}}, z_3 = \frac{9}{\bar{z_3}}$

$|z_1 + z_2 + z_3| = |\frac{1}{\bar{z_1}} + \frac{4}{\bar{z_2}} + \frac{9}{\bar{z_3}}|$

$1 = \frac{|z_2z_3 + 4z_1z_3 + 9z_1z_2|}{|z_1||z_2||z_3|}$

$|z_2z_3 + 4z_1z_3 + 9z_1z_2| = |z_1||z_2||z_3|$

$= 1 \times 2 \times 3 = 6$

9.If z_1, z_2 and z_3 are three complex number S.T. $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$.
Soln:

Given $|z_1| = |z_2| = |z_3| = r$ ∵ $|z|^2 = z\bar{z}$

$z_1\bar{z_1} = z_2\bar{z_2} = z_3\bar{z_3} = r^2$

$z_1 = \frac{r^2}{\bar{z_1}}, z_2 = \frac{r^2}{\bar{z_2}}, z_3 = \frac{r^2}{\bar{z_3}}$

$|z_1 + z_2 + z_3| = |\frac{r^2}{\bar{z_1}} + \frac{r^2}{\bar{z_2}} + \frac{r^2}{\bar{z_3}}|$

$= r^2 |\frac{\bar{z_1}\bar{z_2} + \bar{z_2}\bar{z_3} + \bar{z_3}\bar{z_1}}{\bar{z_1}\bar{z_2}\bar{z_3}}|$

$|z_1 + z_2 + z_3| = r^2 \frac{|z_1z_2+z_2z_3+z_1z_3|}{r^3}$

$= \frac{|z_1z_2+z_2z_3+z_1z_3|}{r}$

∴ $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$

10.Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.
Soln:

Given, $|z| = r = 2$ and

$z_1 = 1 + i\sqrt{3}$;

$\theta = \alpha = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

∴ Euler's form of $z_1 = re^{i\theta} = 2e^{i\frac{\pi}{3}}$

Clearly, z_2 is rotation of z_1 anti-clockwise by $\frac{2\pi}{3}$

$z_2 = z_1e^{i\frac{2\pi}{3}} = 2e^{i\frac{\pi}{3}}e^{i\frac{2\pi}{3}} = 2e^{i\pi} = -2$

Clearly, z_3 is rotation of z_2 anticlockwise by $\frac{2\pi}{3}$

$z_3 = z_2e^{i\frac{2\pi}{3}} = -2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

Note :
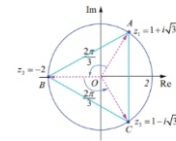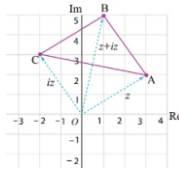
$z = (1)^{\frac{1}{3}} = (1, \omega, \omega^2)$

Here $\omega = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

11.Solve the equation $z^3 + 8i = 0$, where $z \in C$.

Soln: $z^3 + 8i = 0$

$z^3 = -8i$

$$z^3 = (2i)^3 \times 1$$

$$z = 2 \times (1)^{\frac{1}{3}}$$

$$z = 2i(1, \omega, \omega^2)$$

$$z = 2i, \ 2i\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right), \ 2i\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$\mathbf{z = 2i, -i - \sqrt{3}, -i + \sqrt{3}}$$

12.Solve the equation $z^3 + 27 = 0$, where $z \in C$

Soln: $z^3 + 27 = 0$

$$z^3 = -27 = -3 \times -3 \times -3$$

$$z^3 = (-3)^3 \times 1$$

$$z = -3 \times (1)^{\frac{1}{3}}$$

$$z = -3(1, \omega, \omega^2)$$

$\mathbf{Z = -3, -3\omega, -3\omega^2}$ $= -3, 3(-\frac{1}{z} + i\frac{\sqrt{3}}{2}), -3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$

13.If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3 + 8 = 0$ are $-1, 1 - 2\omega, 1 - 2\omega^2$.
Soln:

$$(z-1)^3 + 8 = 0$$

$$(z-1)^3 = -8 = (-2)^3 \times 1$$

$$(z-1) = -2 \times (1)^{\frac{1}{3}}$$

$$z - 1 = -2(1, \omega, \omega^2) = -2, -2\omega, -2\omega^2$$

$$\mathbf{Z = -1, 1 - 2\omega, 1 - 2\omega^2}$$

14.Find all the cube roots of $\sqrt{3} + i$

Soln:

Let $z^3 = re^{i\theta} \Rightarrow z = \left(re^{i\theta}\right)^{\frac{1}{3}}$

$$z = \left(\sqrt{3} + i\right)^{\frac{1}{3}}$$

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2,$$

$$\theta = \alpha = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\sqrt{3} + i = 2cis\left(\frac{\pi}{6}\right)$$

$$\left(\sqrt{3} + i\right)^{\left(\frac{1}{3}\right)} = 2^{\frac{1}{3}}cis\left(\frac{(12k+1)\pi}{18}\right)$$

$k = 0, \quad z = 2^{\frac{1}{3}} cis(\frac{\pi}{18})$

$k = 1, \quad z = 2^{\frac{1}{3}} cis(\frac{13\pi}{18})$

$k = 2, \quad z = 2^{\frac{1}{3}} cis(\frac{25\pi}{18})$

15.If $cos\,\alpha + cos\,\beta + cos\,\gamma = sin\,\alpha + sin\,\beta + sin\,\gamma = 0$,

show that

(i)$cos\,3\,\alpha + cos\,3\,\beta + cos\,3\,\gamma = 3\,cos(\alpha + \beta + \gamma)$
(ii)$sin\,3\,\alpha + sin\,3\,\beta + sin\,3\,\gamma = 3\,sin(\alpha + \beta + \gamma).$
Soln:

$a = \cos\alpha + i\sin\alpha, \ b = \cos\beta + i\sin\beta, \ c = \cos\gamma + i\sin\gamma$

if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3(\cos\gamma + i\sin\gamma)^3$
$\quad = 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$
$= 3[\cos\,(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$(i)\ \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$
$(ii)\ \sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

16.If $2\,cos\,\alpha = x + \frac{1}{x}$ and $2\,cos\,\beta = y + \frac{1}{y}$, show that

(i)$\frac{x}{y} + \frac{y}{x} = 2\,cos(\alpha - \beta)$

(ii) $xy - \frac{1}{xy} = 2i\,sin(\alpha + \beta)$

(iii)$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\,sin(m\alpha - n\beta)$

(iv)$x^m y^n + \frac{1}{x^m y^n} = 2\,cos(m\alpha + n\beta)$

Soln:

Given $x + \frac{1}{x} = 2cos\alpha$

Let $x = cos\alpha + isin\alpha,$

similarly $y = cos\beta + isin\beta$

(i) $\frac{x}{y} = cos(\alpha - \beta) + isin(\alpha - \beta)$

$\quad\quad \frac{y}{x} = cos(\alpha - \beta) - isin(\alpha - \beta)$

$\quad \frac{x}{y} + \frac{y}{x} = 2cos(\alpha - \beta)$

(ii) $xy = cos(\alpha + \beta) + isin(\alpha + \beta)$

$\quad\quad \frac{1}{xy} = cos(\alpha + \beta) - isin(\alpha + \beta)$

$xy - \frac{1}{xy} = 2isin(\alpha + \beta)$

(iii) $\frac{x^m}{y^n} = cos(m\alpha - n\beta) + isin(m\alpha - n\beta)$

$\quad\quad \frac{y^n}{x^m} = cos(m\alpha - n\beta) - isin(m\alpha - n\beta)$

$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2isin(m\alpha - n\beta)$

(iv)$x^m y^n = cos(m\alpha + n\beta) + isin(m\alpha + n\beta)$

$\quad\quad \frac{1}{x^m y^n} = cos(m\alpha + n\beta) - isin(m\alpha + n\beta)$

$$x^m y^n + \frac{1}{x^m y^n} = 2cos(m\alpha + n\beta)$$

2 ,3 Mark

1.Evaluate

Soln:

(i) $i^{1729} = i$

(ii) $i^{-1924} + i^{2018} = i^0 + i^2 = 1 - 1 = 0;$

(iii) $i^{59} + \frac{1}{i^{59}} = i^{59} - i^{59} = 0$

(iv) $i i^2 i^3 \dots . i^{40} = i^{1+2+3+\dots+40}$

$$= i^{\left(\frac{40\times41}{2}\right)} = i^{820} = 1$$

2. If $z_1 = 6 + 7i, \quad z_2 = 3 - 5i$

Soln:

$z_1 + z_2 = (6 + 3) + i(7 - 5) = 9 + 2i$

$z_1 - z_2 = (6 - 3) + i(7 + 5) = 3 + 12i$

$z_1 z_2 = (6 + 7i)(3 - 5i) = 18 - 30i + 21i - 35(-1)$

$\quad = 53 - 9i$

$\frac{z_1}{z_2} = \frac{6+7i}{3-5i} = \frac{-17+51i}{34} = \frac{-17}{34} + \frac{51i}{34}$ $\frac{a+ib}{c+id} = \frac{(ac+bd)+i\,(bc-ad)}{c^2+d^2}$

3.Show that (i) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Soln :

$\frac{19-7i}{9+i} = 2 - i$, $\frac{20-5i}{7-6i} = 2 + i$

$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$z = (2 - i)^{12} + (2 + i)^{12}$

$\bar{z} = (2 + i)^{12} + (2 - i)^{12}$

$\bar{z} = z$, **z is real**

4.Show that (i) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary

Soln:

$\frac{19+9i}{5-3i} = 2 + 3i, \frac{8+i}{1+2i} = 2 - 3i$

$z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$

$z = (2 + 3i)^{15} - (2 - 3i)^{15}$

$\bar{z} = (2 - 3i)^{15} - (2 + 3i)^{15}$

$\bar{z} = -z$

$\therefore z$ **is purely imaginary.**

5. If $z = 3 + 4i$, then find z^{-1}

Soln:

$z^{-1} = \frac{1}{z} = \frac{1}{3+4i} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25}$

$\quad = \frac{3}{25} + \frac{-4i}{25}$ $\frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

6. If $z = (2 + 3i)(1 - i)$, then find z^{-1}

Soln:

$z = 2 - 2i + 3i + 3i(-i) = 2 + i - 3 = -1 + i$

$z^{-1} = \frac{1}{z} = \frac{1}{-1+i} = \frac{-1-i}{(-1)^2+1^2}$

$\quad = \frac{-1-i}{2} = \frac{-1}{2} - \frac{i}{2}$

7.If $z_1 = 3, \quad z_2 = -7i, \ z_3 = 5 + 4i$ show that

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Soln:

$z_2 + z_3 = -7i + (5 + 4i) = 5 - 3i$

$z_1(z_2 + z_3) = 3(5 - 3i) = 15 - 9i - - - - - - -\rightarrow (1)$

$z_1 z_2 + z_1 z_3 = 3(-7i) + 3(5 + 4i) = -21i + 15 + 12i$

$\quad\quad\quad = 15 - 9i - - - - - - - - - - \rightarrow (2)$

$(1),(2) \Rightarrow z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

8.Which one of the point $i, -2 + i$ and 3 is farthest and shortest from the origin?

Soln :

Let $z_1 = i, \quad z_2 = -2 + i, \quad z_3 = 3$

$|z_1| = |i| = \sqrt{1^2} = 1$

$|z_2| = |-2 + i| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$|z_3| = |3| = 3$ Farthest point is 3 and shortest point is i

9. Which one of the point $10 - 8i$, $11 + 6i$ is closest to $1 + i$.

Soln :

Let $z_1 = 10 - 8i, \quad z_2 = 11 + 6i,$ and $z = 1 + i$

$|z_1 - z| = |(10 - 8i) - (1 + i)| = |9 - 9i|$

$\quad\quad\quad = \sqrt{9^2 + (-9)^2} = \sqrt{162}$

$|z_2 - z| = |(11 + 6i) - (1 + i)| = |10 + 5i|$

$\quad\quad\quad = \sqrt{10^2 + 5^2} = \sqrt{125}$

$11 + 6i$ is closest to $1 + i$.

9. If $(1 + i)(1 + 2i)(1 + 3i) \dots\dots(1 + ni) = x + iy$ then show that $2.5.10 \dots\dots(1 + n^2) = x^2 + y^2$.

Soln:

$|(1 + i)(1 + 2i)(1 + 3i) \dots\dots(1 + ni)| = |x + iy|$

$|(1 + i)||(1 + 2i)||(1 + 3i)| \dots\dots|(1 + ni)| = |x + iy|$

$\left(\sqrt{1^2 + 1^2}\right)\left(\sqrt{1^2 + 2^2}\right)\left(\sqrt{1^2 + 3^2}\right)\dots\dots\left(\sqrt{1^2 + n^2}\right)$

$\quad\quad\quad = \sqrt{x^2 + y^2}$

$$(\sqrt{2})(\sqrt{5})(\sqrt{10})\ldots\ldots(\sqrt{1^2+n^2}) = \sqrt{x^2+y^2}$$

Taking square on both sides

$$2.5.10\ldots\ldots(1+n^2) = x^2+y^2$$

Square root of a complex number

If $\quad z = x \pm iy$, then

$$\sqrt{z} = \sqrt{x \pm iy} = \pm\left(\sqrt{\frac{|z|+x}{2}} \pm i\sqrt{\frac{|z|-x}{2}}\right)$$

10. Find the square root of a complex number
$6 - 8i, 4 + 3i$.

Soln:

$$|6 - 8i| = \sqrt{(6)^2+(-8)^2} \ = \sqrt{100}$$

$$|z| = 10$$

$$\sqrt{6-8i} = \pm\left(\sqrt{\frac{10+6}{2}} - i\sqrt{\frac{10-6}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{16}{2}} - i\sqrt{\frac{4}{2}}\right)$$

$$= \pm\left(\sqrt{8} - i\sqrt{2}\right)$$

$$= \pm\left(2\sqrt{2} - i\sqrt{2}\right)$$

$$|4 + 3i| = \sqrt{4^2+3^2} \ = \sqrt{25}$$

$$|z| = 5$$

$$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\sqrt{\frac{5-4}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right)$$

$$= \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

Try yourself :

Find the square root of a complex number $-6 + 8i$,
$-5 - 12i$

11. If area of triangle formed by z, iz, z+iz is 50
sq.unit. find the value of $|z|$.

Soln : Area of triangle $= \frac{1}{2}|z|^2 = 50$

$$|z|^2 = 100 \implies |z| = 10$$

12. If $|z| = 2$ show that $3 \le |z + 3 + 4i| \le 7$

Soln:

$$\left||z| - |3 + 4i|\right| \le |z + 3 + 4i| \le |z| + |3 - 4i|$$

$$|2 - 5| \le |z + 3 + 4i| \le 2 + 5$$

$$|-3| \le |z + 3 + 4i| \le 7$$

$$3 \le |z + 3 + 4i| \le 7$$

13. Find the value of n such that $\left(\frac{1+i}{1-i}\right)^n = 1$

Soln

$$\frac{1+i}{1-i} = i$$

$$\left(\frac{1+i}{1-i}\right)^n = i^n = 1 \qquad \text{Possible values of n is 4,8,12,....}$$

13. Show that $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = -2i$

Soln

$$\frac{1+i}{1-i} = i \ , \ \frac{1-i}{1+i} = -i$$

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3 = -i - i = -2i$$

14.simplify $(i)(1 + i)^{18}$ (ii) $\left(-\sqrt{3} + 3i\right)^{31}$

soln:

(i) $(1 + i)^{18} = ((1 + i)^2)^9$

$$= (2i)^9$$

$$= 512i$$

(ii)$\left(-\sqrt{3} + 3i\right)^{31} = \left[2\sqrt{3}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right]^{31}$

$$= \left(2\sqrt{3}\right)^{31}\omega^{31} = \left(2\sqrt{3}\right)^{31}\omega$$

$$= \left(2\sqrt{3}\right)^{31}\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$$

14.Simplify $\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right]^{30}$

Soln

$$\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right] = cos2\theta + isin2\theta$$

$$\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right]^{30} = (cos2\theta + isin2\theta)^{30}$$

$$= (cos60\theta + isin60\theta)$$

15. Find the locus of z If$|z + i| = |z - 1|$

Soln:

$$|z + i| = |z - 1|$$

$$|x + iy + i| = |x + iy - 1|$$

$$|x + i(y + 1)| = |(x - 1) + iy|$$

$$\sqrt{x^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + y^2}$$

$$x^2 + (y + 1)^2 = (x - 1)^2 + y^2$$

$$x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$$

$$2x + 2y = 0$$

$$x + y = 0$$

16. write $\overline{3i} + \frac{1}{2-i}$ in rectangular form.

Soln:

$$\overline{3i} + \frac{1}{2-i} = -3i + \frac{2+i}{5}$$

$$= \frac{2-14i}{5} = \frac{2}{5} - \frac{14i}{5}$$

$\frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

17.Find $\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

Soln:

$$\left|\frac{i(2+i)^3}{(1+i)^2}\right| = \frac{1\left(\sqrt{2^2+1^2}\right)^3}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{\left(\sqrt{5}\right)^3}{\left(\sqrt{2}\right)^2} = \frac{5\sqrt{5}}{2}$$

18.The complex numbers u,v and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$**. If** $v = 3 - 4i$ **and** $w = 4 + 3i$**, Find** u **in rectangular form.**

Soln:

$$\frac{1}{v} = \frac{1}{3-4i} = \frac{3+4i}{25}$$

$hint: \quad \frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

$$\frac{1}{w} = \frac{1}{4+3i} = \frac{4-3i}{25}$$

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{7+i}{25}$$

$$u = \frac{25}{7+i} = \frac{25(7-i)}{50} = \frac{7}{2} - \frac{i}{2}$$

19.Show that $|3z - 5 + i| = 4$ **represents a circle, then find its centre and radius.**

Soln:

$$|3z - (5 - i)| = 4$$

$$\left|z - \left(\frac{5}{3} - \frac{i}{3}\right)\right| = \frac{4}{3}$$

centre $\left(\frac{5}{3}, -\frac{1}{3}\right)$ radius $\frac{4}{3}$

Hint: $|z - z_0| = r$

Try yourself

(i)$|z + 2 - i| < 2$,(ii) $|z - 2 - i| = 3$ (iii) $|2z + 2 - 4i| = 2$ (iv) $|3z - 6 + 12i| = 8$

20. If $\omega \neq 1$ **is a cube root of unity , show that**

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$$

Soln:

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} \times \frac{\omega}{\omega} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} \times \frac{\omega^2}{\omega^2}$$

$$= \frac{\omega(a + b\omega + c\omega^2)}{a + b\omega + c\omega^2} + \frac{\omega^2(a + b\omega + c\omega^2)}{a + b\omega + c\omega^2}$$

$$= \omega + \omega^2 = -1$$

21.Find the fourth roots of unity.

Soln:

Given $z^4 = 1$

$$(z^2)^2 = 1$$

$$z^2 = \pm\sqrt{1}$$

$$z^2 = \pm 1$$

$z^2 = 1$	$z^2 = -1$
$z = \pm\sqrt{1}$	$z = \pm\sqrt{-1}$
$z = \pm 1$	$z = \pm i$

22.Find the cube roots of unity.

Soln:

Given $z^3 = 1$

$$z^3 - 1 = 0$$

$$(z - 1)(z^2 + z + 1) = 0$$

$z - 1 = 0$	$z^2 + z + 1 = 0$
$z = 1$	$z = \frac{-1 \pm i\sqrt{3}}{2}$

23.Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$

Soln:

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$$

$$= -i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

22. Evaluate $\sum_{k=1}^{18}(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9})$

Soln:

$$\sum_{k=0}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = 0$$

$$1 + \sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = 0$$

$$\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = -1$$

23. If $\omega \neq 1$ **is a cube root of unity, then show that the following**

(i) $\left(1 - \omega + \omega^2\right)^6 + \left(1 + \omega - \omega^2\right)^6 = 128$

(ii) $(1 + \omega)\left(1 + \omega^2\right)\left(1 + \omega^4\right).....\left(1 + \omega^{2^{11}}\right) = 1$

Soln:

(i) $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6$

$$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6$$

$$= (-2\omega)^6 + (-2\omega^2)^6$$

$$= 64 + 64 = 128$$

(ii) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8).....\left(1 + \omega^{2^{11}}\right)$

$$= [(-\omega^2)(-\omega)][(-\omega^2)(-\omega)]...upto\ 6\ times$$

$$= 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1$$

24.State and prove Triangle inequality

Triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$

Proof:

$OA = |z_1|, OB = |z_2|, OC = |z_1 + z_2|$

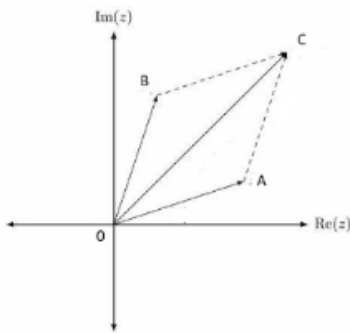In Δ OAC, $\qquad OC < OA + AC$

$|z_1 + z_2| < |z_1| + |z_2| - - - - - (1)$

Suppose the points are in collinear

$|z_1 + z_2| = |z_1| + |z_2| - - - - \rightarrow (2)$

From (1),(2)

$|z_1 + z_2| \leq |z_1| + |z_2|$

DISCRETE MATHEMATICS

5 MARK

Important hints:

Let * be a binary operation on S

i) Closure property : $\forall a, b \in S \Rightarrow a * b \in S$

ii) Commutative property : $a * b = b * a$, $\forall a, b \in S$

iii) Associative property :

$$a * (b * c) = (a * b) * c \; , \forall a, b, c \in S$$

iv) Existence of identity : $a * e = e * a = a$, e is the

identity element, $e \in S, \forall a \in S$

v) Existence of inverse : a^{-1} is the inverse of a

$$a * \quad a^{-1} = a^{-1} * a = e, \quad a^{-1} \in S$$

1.Verify closure, commutative, associative, existence of identity, and existence of inverse for $m * n = m + n - mn, \; m, n \epsilon Z$
Soln:

Closure property:

$m, n \epsilon Z, \quad$ clearly $m + n - mn \; \epsilon Z$

$\therefore$ closure property true

Associative property:

$(l * m) * n = l * (m * n)$

$(l * m) * n = l + m + n - lm - mn - nl + lmn$

$\qquad = l * (m * n)$

$\therefore$ associative property true

Identity property:

$m * e = e * m = m$

$m + e - me = m$

$e = 0 \; \epsilon Z$

$\therefore$ identity property true

Inverse property:

$m * m^{-1} = m^{-1} * m = e = 0$

$m^{-1} = \frac{-m}{1-m} \notin z$

$\therefore$ inverse property not true

Commutative property:

$m * n = n * m = m + n - mn = n + m - nm$

$\therefore$ commutative property true

2.Let A be $Q \setminus \{1\}$. **Define* on A by** $x * y = x + y - xy$, **Verify closure, commutative, associative, existence of identity, and existence of inverse properties satisfied by * on A.**
Soln:

Closure property:

$x, y \epsilon Q \setminus \{1\}, \quad x \neq 1, \; y \neq 1$

$\Rightarrow \; x + y - xy \neq 1$

$x * y \epsilon Q \setminus \{1\} \; \therefore$ closure property true

Associative property:

$(x * y) * z = x * (y * z)$

$\therefore$ associative property true

Identity property:

$x * e = e * x = x$

$e = 0 \; \epsilon Q \setminus \{1\}$

$\therefore$ identity property true

Inverse property:

$x * x^{-1} = x^{-1} * x = e = 0$

$x^{-1} = \frac{-x}{1-x} \epsilon Q \setminus \{1\}$

$\therefore$ inverse property true

Commutative property:

$x * y = x + y - xy = y + x - yx = y * x$

$\therefore$ commutative property true

3.Verify closure, commutative , associative, existence of identity, and inverse for

$M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in R - \{0\} \right\}$. * be matrix multiplication

Soln:

Let * **be the matrix multiplication.**

Closure property:

Let, $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ $\because$ $x, y \neq 0 \Rightarrow 2xy \neq 0$

$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \epsilon M$

$\therefore$ closure property true

Commutative property :

$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$, $BA = \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$

    $AB = BA$

$\therefore$ commutative property true.

Associative property:

Matrix multiplication always satisfies associative property

Existence of identity property: $A * E = E * A = A$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

        $2ex = x$

$e = \frac{1}{2}$ $\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \epsilon M$

$\therefore$ identity property true

Existence of inverse property:

$A * A^{-1} = A^{-1} * A = E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

        $2ax = \frac{1}{2}$ $\Rightarrow a = \frac{1}{4x}$

    $\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \epsilon M$

$\therefore$ inverse property true

4.Verify closure property, commutative property, associative property ,existence of identity, and existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$.
Soln:

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

Closure property:

From the table closure property true.

Commutative property:

From the table commutative property true.

Associative property:

$\times_{11}$ always satisfies associative property .

Identity property:

Identity element $1 \epsilon A$

$\therefore$ identity property true

Inverse property:

Inverse element of 1,3,4,5 and 9 are 1,4,3,9 and 5 respectively.

$\therefore$ inverse property true

5.Verify closure property, commutative property, associative property,existence of identity, and existence of inverse for the operation $+_5$ on Z_5 using table corresponding to addition modulo 5.
Soln:

$Z_5 = \{0,1,2,3,4\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Closure property:

From the table closure property true

Commutative property:

From the table commutative property true

Associative property:

$+_5$ always satisfies associative property.

Identity property:

identity element $0 \epsilon Z_5$

$\therefore$ identity property true.

Inverse property:

Inverse element of 0,1,2,3 and 4 are 0,4,3,2 and 1 respectively.

∴ inverse property true

6.Verify closure, commutative, associative, identity, and inverse property for

$a * b = \frac{a+b}{2} \ \forall \ a, b \ \epsilon \ Q$

Soln:

Closure property:

Clearly $a, b \ \epsilon \ Q \implies \frac{a+b}{2} \ \epsilon \ Q$

∴ closure property true

Associative property:

$(a * b) * c = \frac{a+b+2c}{4}$

$a * (b * c) = \frac{2a+b+c}{4}$

$(a * b) * c \neq a * (b * c)$

∴ Associative property is not true

Identity property:

$a * e = e * a = a$

$\quad a * e = a$

$\quad \frac{a+e}{2} = a$

$\quad\quad e = a$

Uniqueness of identity is not preserved

∴ identity property is not true

Inverse property:

∴ inverse property is not true

Commutative property:

$a * b = b * a = \frac{a+b}{2}$

Try Yourself:

Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ be any three Boolean matrices of the same type.

Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

4.Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

∵commutative property true

2,3 Marks

1.In an algebraic structure the identity element must be unique

Soln:

Let e_1 and e_2 be the identity element of S

$\quad a * e_1 = e_1 * a = a \quad\quad \forall a \in S$

$\quad a * e_2 = e_2 * a = a \quad\quad \forall a \in S$

$\quad\quad a * e_1 = a * e_2$

$\quad\quad\quad \therefore e_1 = e_2$

2.In an algebraic structure the inverse element must be unique

Soln:

Let a_1 and a_2 be the inverse element of a in S

$\quad a * a_1 = a_1 * a = e \quad\quad \forall a \in S$

$\quad a * a_2 = a_2 * a = e \quad\quad \forall a \in S$

$\quad\quad a * a_1 = a * a_2$

$\quad\quad\quad a_1 = a_2$

3. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

Soln:

$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

5. Show that $p \leftrightarrow q \equiv (p \wedge q)\vee(\neg q \wedge \neg p)$

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg q \wedge \neg p$	$(p \wedge q)\vee(\neg q \wedge \neg p)$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T

$$p \leftrightarrow q \equiv (p \wedge q)\vee(\neg q \wedge \neg p)$$

6. Show that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee(\neg q \vee r)$

P	q	r	$\neg p$	$\neg q$	$\neg q \vee r$	$p \rightarrow (\neg q \vee r)$	$\neg p \vee(\neg q \vee r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$$p \rightarrow (\neg q \vee r) \equiv \neg p \vee(\neg q \vee r)$$

7. Show that $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

$$p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

8.Using truth table whether the statements $\neg(p \vee q)\vee(\neg p \wedge q)$ and $\neg p$ are logically equivalent.

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p	q	¬p	p∨q	¬(p∨q)	¬p ∧ q	¬(p∨q)∨(¬p ∧ q)
T	T	F	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	F	T	T
F	F	T	F	T	F	T

$\neg(p\lor q)\lor(\neg p\land q)$ and $\neg p$ are logically equivalent.

9.Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

10. Show that $p \rightarrow q$ **and** $q \rightarrow p$ **are not equivalent**

P	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \rightarrow q$ and $q \rightarrow p$ are not equivalent

11. Show that q$\rightarrow p \equiv \neg p \rightarrow \neg q$

P	q	q→ p	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

q→ $p \equiv \neg p \rightarrow \neg q$

12.Prove that $\neg(p \land q) \equiv \neg p \lor \neg q$

P	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\neg(p \land q) \equiv \neg p \lor \neg q$

13.Verify whether the compound propositions are tautology or contradiction or contingency.

$$\big((p \lor q) \land \neg p\big) \to q$$

P	q	$p \lor q$	$\neg p$	$(p \lor q) \land \neg p$	$\big((p \lor q) \land \neg p\big) \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

$\big((p \lor q) \land \neg p\big) \to q$ is a tautology.

14.Construct the truth table for $\big(p\overline{\lor}q\big) \land (p\overline{\lor}\neg q)$

P	q	$p\overline{\lor}q$	$\neg q$	$p\overline{\lor}\neg q$	$\big(p\overline{\lor}q\big) \land (p\overline{\lor}\neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	T	F	F	F
F	F	F	T	T	F

15. show that without using truth table

$p \to (q \to r) \equiv (p \land q) \to r$

Soln:
$$p \to (q \to r) \equiv \neg p \lor (q \to r)$$
$$\equiv \neg p \lor (\neg q \lor r)$$
$$\equiv (\neg p \lor \neg q) \lor r$$
$$\equiv \neg(p \land q) \lor r$$
$$\equiv (p \land q) \to r$$
$$p \to (q \to r) \equiv (p \land q) \to r$$

16.Show that without using truth table

$p \leftrightarrow q \equiv (p \land q) \lor (\neg q \land \neg p)$

Soln:
$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$
$$\equiv (\neg p \lor q) \land (p \lor \neg q)$$
$$\equiv (\neg p \land (p \lor \neg q) \lor ([q \land (p \lor \neg q)])$$
$$\equiv [(\neg p \land p) \lor (\neg p \land \neg q)] \lor [(q \land p) \lor (q \land \neg q)]$$
$$\equiv [F \lor (\neg p \land \neg q)] \lor [(q \land p) \lor F]$$
$$\equiv (q \land p) \lor (\neg p \land \neg q)$$
$$p \leftrightarrow q \equiv (p \land q) \lor (\neg q \land \neg p)$$

17. check whether the $p \to (q \to p)$ is tautology or contradiction without using truth table.

Soln:
$$p \to (q \to p) \equiv \neg p \lor (q \to p)$$
$$\equiv \neg p \lor (\neg q \lor p)$$
$$\equiv \neg p \lor (p \lor \neg q)$$
$$\equiv (\neg p \lor p) \lor \neg q$$
$$\equiv T \lor \neg q$$
$$\equiv T \therefore p \to (q \to p) \text{ is tautology}$$

DIFFERENTIALS AND PARTIAL
DERIVATIVES

Important hints:

linear approximation :

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Euler theorem : $\qquad x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$
$Degree = n = N.degree - D.degree$

1.Find the linear approximation for
f(x)= $\sqrt{1+x}, x \geq -1, at\ x_0 = 3$ Use the linear
approximation to estimate $f(3.2)$
Soln

$f(x) = \sqrt{1+x}$, $x_0 = 3, \Delta x = 0.2$ and

hence $f(3) = \sqrt{1+3} = 2$.

$f'(x) = \dfrac{1}{2\sqrt{1+x}} \Rightarrow f'(3) = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\qquad = 2 + \dfrac{1}{4}(x-3) = \dfrac{x}{4} + \dfrac{5}{4}$

$f(3.2) = \sqrt{4.2} \cong L(3.2) = \dfrac{3.2}{4} + \dfrac{5}{4} = 2.050$

2.Use linear approximation to find an
approximate value of $\sqrt{9.2}$ without using a
calculator
Soln :

$f(x) = \sqrt{x}, x_0 = 9, \Delta x = 0.2$

$\qquad f(9) = 3,$

$f'(x) = \dfrac{1}{2\sqrt{x}}$, $\quad f'(x) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{(2\times3)} = \dfrac{1}{6}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt{9.2} = f(9) + f'(9)(x - 9)$

$\qquad = 3 + \dfrac{1}{6}(9.2 - 9) = 3 + \dfrac{0.2}{6} = 3.0333$

3.If $u = sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right), show\ that$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}tan\ u$

Soln:

$u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$f = \sin u = \left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$Degree = n = N.degree - D.degree$

$\qquad n = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$\therefore f(x,y)$ is a homgeneous function of degree is

$$n = \dfrac{1}{2}$$

$By\ Euler\ theorem, \qquad x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

$x\dfrac{\partial}{\partial x}(\sin u) + y\dfrac{\partial}{\partial y}(\sin u) = \dfrac{1}{2}\sin u$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$

4.If $u(x,y) = \dfrac{x^2+y^2}{\sqrt{x+y}}, prove\ that\ x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}u.$

Soln:

$u(x,y) = \dfrac{x^2+y^2}{\sqrt{x+y}}$

$Degree = n = N.degree - D.degree$

$\qquad n = 2 - \dfrac{1}{2} = \dfrac{3}{2}$

$\therefore u(x,y)$ is a homgeneous function of degree is

$\qquad n = \dfrac{3}{2}$

$By\ Euler\ theorem, \quad x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

$\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}u$

5.If $v(x,y) = \log\left(\dfrac{x^2+y^2}{x+y}\right), prove\ that\ x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 1$

Soln

$v(x,y) = \log\left(\dfrac{x^2+y^2}{x+y}\right)$

$f = e^v = \dfrac{x^2+y^2}{x+y}$

$Degree = n = N.degree - D.degree$

$\qquad n = 2 - 1 = 1$

$\therefore f(x,y)$ is a homgeneous function of degree is

$\qquad n = 1$

$By\ Euler\ theorem, \quad x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

$\qquad x\dfrac{\partial e^v}{\partial x} + y\dfrac{\partial e^v}{\partial y} = (1)e^v$

$\qquad x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 1$

6.If w(x,y,z)= $\log\left(\dfrac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right), find$

$x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z}$

Soln

$$w(x,y,z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$$

$$f = e^w = \left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$$

$Degree = n = N.\,degree - D.\,degree$

$$n = 7 - 2 = 5$$

$\therefore f(x,y,z)$ is a homgeneous function of degree is

$$n = 5$$

$By\ Euler\ theorem, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$

$$x\frac{\partial e^w}{\partial x} + y\frac{\partial e^w}{\partial y} + z\frac{\partial e^w}{\partial z} = (5)e^w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

7.Prove that $g(x,y) = xlog\left(\frac{y}{x}\right)$ is homogeneous,

verify Euler's theorem for g

Soln

$$g(x,y) = xlog\left(\frac{y}{x}\right)$$

$$Degree = n = N.\,degree - D.\,degree$$

$$n = 2 - 1 = 1$$

$\therefore f(x,y)$ is a homgeneous function of degree is

$$n = 1$$

$By\ Euler\ theorem, x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 1g$

$LHS = x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\frac{\partial}{\partial x}\left(xlog\frac{y}{x}\right) + y\frac{\partial}{\partial y}\left(xlog\frac{y}{x}\right)$

$$= x\log\frac{y}{x} = g$$

$Hence\ x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 1g$

THEORY OF EQUATIONS

❖ $ax^3 + bx^2 + cx + d = 0$

❖ Sum of co-efficients = 0

  ⇒ $x = 1$ is a root

❖ Sum of co-efficients $a + c = b + d$

  ⇒ $x = -1$ is a root

❖ Otherwise try $x = 2$ or 3 is a root

<u>1.</u>Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

Soln:

Given $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$$

$$y^2 - 2 - 10y + 26 = 0$$

$$y^2 - 10y + 24 = 0$$

$$(y - 6)(y - 4) = 0$$

$$y = 6, \quad y = 4$$

Case(i) $x + \frac{1}{x} = 6$	Case(ii) $x + \frac{1}{x} = 4$
$\frac{x^2+1}{x} = 6$	$\frac{x^2+1}{x} = 4$
$x^2 + 1 = 6x$	$x^2 + 1 = 4x$
$x^2 - 6x + 1 = 0$	$x^2 - 4x + 1 = 0$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= 3 \pm 2\sqrt{2}$	$= 2 \pm \sqrt{3}$

2.Solve $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

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Soln:

$$\begin{array}{c|ccccc} 2 & 6 & -35 & 62 & -35 & 6 \\ & 0 & 12 & -46 & 32 & -6 \\ \hline 3 & 6 & -23 & 16 & -3 & \boxed{0} \\ & 0 & 18 & -15 & 3 & \\ \hline & 6 & -5 & 10 & \boxed{0} & \end{array}$$

Reduced equation is

$$6x^2 - 5x + 1 = 0$$

$$\left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right) = 0$$

$$x = 2, 3, \frac{1}{2}, \frac{1}{3}$$

3.Solve $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ **whose one of the roots is** $\frac{1}{3}$ **then find the other roots   M-23**

Soln:

$$\begin{array}{c|ccccc} \frac{1}{3} & 6 & -5 & -38 & -5 & 6 \\ & 0 & 2 & -1 & -13 & -6 \\ \hline 3 & 6 & -3 & -39 & -18 & \boxed{0} \\ & 0 & 18 & 45 & 18 & \\ \hline & 6 & 15 & 6 & \boxed{0} & \end{array}$$

Reduced equation is

$$6x^2 + 15x + 6 = 0$$

$$\left(x + \frac{12}{6}\right)\left(x + \frac{3}{6}\right) = 0$$

$$(x + 2)\left(x + \frac{1}{2}\right) = 0$$

$$x = \frac{1}{3}, 3, -2, -\frac{1}{2}$$

4.Solve $x^4 + 3x^3 - 3x - 1 = 0$

Soln:

$$\begin{array}{c|ccccc} 1 & 1 & 3 & 0 & -3 & -1 \\ & 0 & 1 & 4 & 4 & 1 \\ \hline -1 & 1 & 4 & 4 & 1 & \boxed{0} \\ & 0 & -1 & -3 & -1 & \\ \hline & 1 & 3 & 1 & \boxed{0} & \end{array}$$

Reduced equation is

$$x^2 + 3x + 1 = 0$$

$$x = \frac{-3 + \sqrt{5}}{2}, \quad x = \frac{-3 - \sqrt{5}}{2}$$

$$x = 1, -1, \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$$

5.If $2 + i$ **and** $3 - \sqrt{2}$ **are the roots of the equation** $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$ **find all roots.**

Soln: Given roots $2 + i$, $3 - \sqrt{2}$

Other roots $2 - i$, $3 + \sqrt{2}$

Let assume missing roots a **and** b.

S.R: $2 + i$ + $3 - \sqrt{2}$ + $2 - i$ + $3 + \sqrt{2} + a + b = 13$

$$10 + a + b = 13$$

$$a + b = 3$$

P.R : $(2 + i)(2 - i)(3 - \sqrt{2})(3 + \sqrt{2}) \, ab = -140$

$$5(7)ab = -140$$

$$ab = \frac{-140}{35} = -4$$

Reduced equation $x^2 - 3x - 4 = 0$

$$(x - 4)(x + 1) = 0$$

$$x = 4, \quad x = -1$$

6.If $1 + 2i$ **and** $\sqrt{3}$ **are the roots of the equation** $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135 = 0$ **find all roots.**

Soln: **Given roots** $1 + 2i$, $\sqrt{3}$

then other roots $1 - 2i$, $-\sqrt{3}$

Let assume missing roots a **and** b.

SR: $1 + 2i$ + $\sqrt{3}$ + $1 - 2i$ + $(-\sqrt{3}) + a + b = 3$

$$a + b = 1$$

PR: $(1 + 2i)(1 - 2i)(\sqrt{3})(-\sqrt{3}) \, ab = 135$

$$5(-3) \, ab = 135$$

$$ab = \frac{135}{-15} = -9$$

Reduced equation $x^2 - x - 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1^2 - 4 \times 1 \times -9}}{2 \times 1} = \frac{1 \pm \sqrt{37}}{2}$$

7. Solve the eqn $3x^3 - 16x^2 + 23x - 6 = 0$ **if the product of two roots is 1.**

Soln:

Let a,b,c are roots of $x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - \frac{6}{3} = 0$

$$abc = 2$$

$$c = 2$$

$$\begin{array}{c|cccc} 2 & 3 & -16 & 23 & -6 \\ & 0 & 6 & -20 & 6 \\ \hline & 3 & -10 & 3 & \boxed{0} \end{array}$$

Reduced equation

$$3x^2 - 10x + 3 = 0$$

$$\left(x - \frac{9}{3}\right)\left(x - \frac{1}{3}\right) = 0 \qquad x = 2, 3, \frac{1}{3}$$

8. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ **if it is given that two of its roots are in the ratio 3:2**

Soln:

$$\begin{array}{c|cccc} -1 & 1 & -9 & 14 & 24 \\ & 0 & -1 & 10 & -24 \\ \hline & 1 & -10 & 24 & \boxed{0} \end{array}$$

Reduced equation

$x^2 - 10x + 24 = 0$

$(x - 6)(x - 4) = 0$

$$x = -1, 4, 6$$

Try yourself:

Solve $2x^3 + 11x^2 - 9x - 18 = 0$ J-23

9.Solve $2x^3 - 9x^2 + 10x = 3$ M-22

Soln: $2x^3 - 9x^2 + 10x - 3 = 0$

1	2	-9	10	-3
	0	2	-7	3
	2	-7	3	0

Reduced equation is

$$2x^2 - 7x + 3 = 0$$

$$\left(x - \frac{6}{2}\right)\left(x - \frac{1}{2}\right) = 0$$

$$x = 1, 3, \frac{1}{2}$$

10.Solve the equation $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.

Soln: Let a, b, c be the roots of

$x^3 - \frac{1}{2}x^2 - \frac{18}{2}x + \frac{9}{2} = 0$

$\frac{1}{2}$	2	-1	-18	9
	0	1	0	-9
	2	0	-18	0

$a + b = 0$

$a + b + c = \frac{1}{2} = c = \frac{1}{2}$

Reduced equation is $2x^2 - 18 = 0$

$x = 3, \quad x = -3$

$$x = \frac{1}{2}, 3, -3$$

11.Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an A.P.

Soln: Let $a - b, a, a + b$ be the roots of

$x^3 - \frac{36}{9}x^2 + \frac{44}{9}x - \frac{19}{9} = 0$

$\frac{4}{3}$	9	-36	44	-16
	0	12	-32	16
	9	-24	12	0

$a - b + a + a + b = 4$

$$a = \frac{4}{3}$$

Reduced equation is $9x^2 - 24x + 12 = 0$

$$\left(x - \frac{18}{9}\right)\left(x - \frac{6}{9}\right) = 0$$

$$x = 2, \frac{2}{3}, \frac{4}{3}$$

12.solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if the roots form an G.P.

Soln:

Let $ar, a, \frac{a}{r}$ be the roots of $x^3 - \frac{26}{3}x^2 + \frac{52}{3}x - \frac{24}{3} = 0$

$a^3 = 8 \quad \Rightarrow \quad a = 2$

2	3	-26	52	-24
	0	6	-40	24
	3	-20	12	0

Reduced equation is

$3x^2 - 20x + 12 = 0; \left(x - \frac{18}{3}\right)\left(x - \frac{2}{3}\right) = 0$

$$x = 2, 6, \frac{2}{3}$$

13.Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one root is twice the sum of the other two roots.

Soln:

Let a, b, c be the roots of :

Given $a = 2(b + c)$

$a + b + c = 3$

$2a + 2b + 2c = 6$

$3a = 6$

$a = 2$

2	2	-6	3	k
	0	4	-4	-2
	2	-2	-1	k-2

Reduced equation is $; k - 2 = 0$

$2x^2 - 2x - 1 = 0$ $; k = 2$

$x = \frac{1 \pm \sqrt{3}}{2}$ $x = 2, \frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}$

14.Solve $(x - 2)(x - 7)(x - 3)(x + 2) + 19 = 0$

Soln: $(x - 2)(x - 3)(x - 7)(x + 2) + 19 = 0$

$\left(x^2 - 5x + 6\right)\left(x^2 - 5x - 14\right) + 19 = 0$

Put $x^2 - 5x = y$ $(y + 6)(y - 14) + 19 = 0$

$y^2 - 8y - 84 + 19 = 0$

$y^2 - 8y - 65 = 0$

$y = 13, \qquad y = -5$

Case(i) $y = 13$	Case(ii) $y = -5$
$x^2 - 5x = 13$	$x^2 - 5x = -5$

$x^2 - 5x - 13 = 0$	$x^2 - 5x + 5 = 0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{77}}{2}$	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{5}}{2}$

15. Solve $(2x-3)(6x-1)(3x-2)(x-2) - 5 = 0$

Soln: $(2x-3)(3x-2)(6x-1)(x-2) - 5 = 0$

$\left(6x^2 - 13x + 6\right)\left(6x^2 - 13x + 2\right) - 5 = 0$

Put $6x^2 - 13x = y$ $(y+6)(y+2) - 5 = 0$

$y^2 + 8y + 7 = 0$

$y = -1, \qquad y = -7$

Case(i) $y = -1$	Case(ii) $y = -7$
$6x^2 - 13x = -1$	$6x^2 - 13x = -7$
$6x^2 - 13x + 1 = 0$	$6x^2 - 13x + 7 = 0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$	$x = 1, \frac{7}{6}$
$x = \frac{13 \pm \sqrt{145}}{12}$	

16. Solve $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$

Soln: $(2x-1)(2x+3)(x+3)(x-2) + 20 = 0$

$\left(4x^2 + 4x - 3\right)\left(x^2 + x - 6\right) + 20 = 0$

put $x^2 + x = y$ $(4y-3)(y-6) + 20 = 0$

$4y^2 - 27y + 18 + 20 = 0$

$4y^2 - 27y + 38 = 0$

$y = \frac{19}{4}, \quad y = \frac{8}{4} = 2$

Case(i) $y = \frac{19}{4}$	Case(ii) $y = 2$
$x^2 + x = \frac{19}{4}$	$x^2 + x = 2$
$4x^2 + 4x - 19 = 0$	$x^2 + x - 2 = 0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$	$x = -2, \qquad x = 1$
$= \frac{-1 \pm 2\sqrt{5}}{2}$	

17. Find the polynomial equation

(i) $2 + i\sqrt{3}$ (ii) $2i + 3$ (iii) $\sqrt{5} - \sqrt{3}$

(iv) $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ (v) $2 - \sqrt{3}$ M-22

Soln:

(i) $x = 2 + i\sqrt{3}$	(ii) $x = 2i + 3$
$x - 2 = i\sqrt{3}$	$x - 3 = 2i$
$(x-2)^2 = \left(i\sqrt{3}\right)^2$	$(x-3)^2 = (2i)^2$
$x^2 - 4x + 4 = -3$	$x^2 - 6x + 9 = -4$
$x^2 - 4x + 7 = 0$	$x^2 - 6x + 13 = 0.$

(iii) $x = \sqrt{5} - \sqrt{3}$	(iv) $x = \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$
$x^2 = \left(\sqrt{5} - \sqrt{3}\right)^2$	$x^2 = \frac{\sqrt{2}}{\sqrt{3}}$
$x^2 = 5 + 3 - 2\sqrt{15}$	
$x^2 - 8 = -2\sqrt{15}$	$(x^2)^2 = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2$
$\left(x^2 - 8\right)^2 = \left(-2\sqrt{15}\right)^2$	$x^4 = \frac{2}{3}$
$x^4 - 16x^2 + 64 = 60.$	$3x^4 = 2 \text{(or)} 3x^4 - 2 = 0$
$x^4 - 16x^2 + 4 = 0$	

18. Discuss the nature of the roots of equation

(i) $9x^9 + 2x^5 - x^4 - 7x^2 + 2 = 0$

fun	Signs	No.of changes	No.of Real roots
$f(x)$	+ + - - +	2	2 +Ve
$f(-x)$	- - - - +	1	1 -Ve

No. of Imaginary roots = 9 - 3 = 6

(ii) $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x = 0$ J-23

$x\left(x^8 + 9x^6 + 7x^4 + 5x^2 + 3\right) = 0$

$x = 0$ is a root with multiplicity one

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fun	Signs				No.of changes	No.of Real roots	
$f(x)$	+	+	+	+	+	0	0 +Ve
$f(-x)$	-	-	-	-	-	0	0 -Ve

No. of Imaginary roots = 9-1 = 8

(iii) $x^9 - 5x^8 - 14x^7 = 0 \Rightarrow x^7(x^2 - 5x - 14) = 0$

$x = 0$ is a root with multiplicity seven

Fun	Signs			No.of changes	No.of Real roots	
$f(x)$	+	-	-	1	1 +Ve
$f(-x)$	-	-	+	1	1 -Ve

No. of Imaginary roots = 9 - 9 = 0

19. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid. (S-21)

Soln: $(x + 1)(x + 2)(x + 3) - x^3 = 52$

$$x^3 + 6x^2 + 11x + 6 - x^3 = 52$$
$$6x^2 + 11x + 6 = 52$$
$$6x^2 + 11x - 46 = 0$$
$$x = \frac{12}{6}, x = \frac{-23}{6}$$
$$x = 2, \quad x = \frac{-23}{6}(not\ possible)$$

∴ The volume of the cuboid $= (x + 1)(x + 2)(x + 3)$

$$= 3 \times 4 \times 5 = 60$$

20. Construct a cubic equation with roots

(i) 1, 2 , and 3

$$(x - 1)(x - 2)(x - 3) = 0$$
$$x^3 - 6x^2 + 11x - 6 = 0$$

(ii) 1,1, and −2

$$(x - 1)(x - 1)(x + 2) = 0$$
$$x^3 - 0x^2 - 3x + 2 = 0$$

(iii) 2 ,$\frac{1}{2}$, and 1.

$$(x - 2)\left(x - \frac{1}{2}\right)(x - 1) = 0$$
$$x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$$

$$2x^3 - 7x^2 + 7x - 2 = 0$$

21. If α , β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are (i) 2α , 2β , 2γ (ii) $\frac{1}{\alpha}$,$\frac{1}{\beta}$,$\frac{1}{\gamma}$

(iii) $-\alpha, -\beta, -\gamma$ (iv) $\frac{\alpha}{2}$,$\frac{\beta}{2}$,$\frac{\gamma}{2}$

Soln:

(i) 2α , 2β , 2γ

$$2^0 x^3 + 2^1 2x^2 + 2^2 3x + 2^3 4 = 0$$
$$x^3 + 4x^2 + 12x + 32 = 0$$

(ii) $\frac{1}{\alpha}$,$\frac{1}{\beta}$,$\frac{1}{\gamma}$

$$4x^3 + 3x^2 + 2x + 1 = 0$$

(iii) $-\alpha, -\beta, -\gamma$

$$-x^3 + 2x^2 - 3x + 4 = 0$$
$$x^3 - 2x^2 + 3x - 4 = 0$$

(iv) $\frac{\alpha}{2}$,$\frac{\beta}{2}$,$\frac{\gamma}{2}$

$$\left(\frac{1}{2}\right)^0 x^3 + \left(\frac{1}{2}\right)^1 2x^2 + \left(\frac{1}{2}\right)^2 3x + \left(\frac{1}{2}\right)^3 4 = 0$$
$$x^3 + x^2 + \frac{3}{4}x + \frac{4}{8} = 0$$
$$8x^3 + 8x^2 + 6x + 4 = 0$$

22. If α , β , γ and δ are the roots of polynomial eqn $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$.

Soln: Given $2x^4 + 5x^3 - 7x^2 + 8 = 0$

$$x^4 + \frac{5}{2}x^3 - \frac{7}{2}x^2 + 0x + \frac{8}{2} = 0$$

$\alpha + \beta + \gamma + \delta = \frac{-5}{2}$; $\alpha\beta\gamma\delta = \frac{8}{2}$

Required equation is $x^2 - (S.R)x + (P.R) = 0$

$$x^2 - \left(\frac{-5}{2} + \frac{8}{2}\right)x + \left(\frac{-5}{2} \times \frac{8}{2}\right) = 0$$
$$x^2 - \frac{3}{2}x - \frac{40}{4} = 0$$
$$2x^2 - 3x - 20 = 0$$

23. If p and q are roots of eqn $lx^2 + nx + n = 0$,

show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$. M-23

Soln: Given, $lx^2 + nx + n = 0$

$$p + q = -\frac{n}{l} \qquad pq = \frac{n}{l} \Rightarrow \sqrt{pq} = \sqrt{\frac{n}{l}}$$

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = \frac{p+q}{\sqrt{pq}} + \sqrt{\frac{n}{l}} = -\sqrt{\frac{n}{l}} + \sqrt{\frac{n}{l}} = 0$$

24. If the equations $x^2 + px + q = 0$ and

$x^2 + p'x + q' = 0$ **have a common root, show that it must be equal to** $\frac{pq'-qp'}{q-q'}$ **or** $\frac{q-q'}{p-p'}$.

Soln: Let us assume 'a' be the common root

$a^2 + pa + q = 0$

$a^2 + p'a + q' = 0$

$$\frac{a^2}{\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}} = \frac{a}{\begin{vmatrix} q & 1 \\ q' & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & p \\ 1 & p' \end{vmatrix}}$$

$$\frac{a^2}{pq'-qp'} = \frac{a}{q-q'} = \frac{1}{p'-p}$$

$$\frac{a^2}{pq'-qp'} = \frac{a}{q-q'} \qquad \frac{a}{q-q'} = \frac{1}{p'-p}$$

$$a = \frac{pq'-qp'}{q-q'} \quad \text{(or)} \qquad a = \frac{q-q'}{p-p'}$$

25.If α, β be the roots of $x^2 - 5x + 6 = 0$,

find $\alpha^2 - \beta^2$

Soln:

$\alpha + \beta = 5, \quad \alpha\beta = 6 \qquad$ S-21

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 5^2 - 4(6) = 1$$

$$\alpha - \beta = \pm 1$$

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 5(\pm 1) = \pm 5$$

26.If α, β be the roots of $x^2 + 5x + 6 = 0$,

find $\alpha^2 + \beta^2$

Soln: $\alpha + \beta = -5, \quad \alpha\beta = 6 \qquad$ J-22

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-5)^2 - 2(6) = 13$$

27.Solve

(i) $x^4 - 3x^2 - 4 = 0$

(ii) $x^4 - 14x^2 + 45 = 0$

Soln:

(i) $x^4 - 3x^2 - 4 = 0$ J-22		(ii) $x^4 - 14x^2 + 45 = 0$	
$x^2 = -1$	$x^2 = 4$	$x^2 = 9$	$x^2 = 5$
$x = \pm\sqrt{-1}$ $= \pm i$	$x = \pm 2$	$x = \pm 3$	$x = \pm\sqrt{5}$

28.Find the condition of the equation

$x^3 + px^2 + qx + r = 0$ **whose roots are in A.P.** S-20

Soln: Let roots are $a - d, a, a + d$

sum of roots $a - d + a + a + d = -p$

$$3a = -p$$

$$a = \frac{-p}{3}$$

$$\left(\frac{-p}{3}\right)^3 + p\left(\frac{-p}{3}\right)^2 + q\left(\frac{-p}{3}\right) + r = 0$$

$$-p^3 + 3p^3 - 9pq + r = 0$$

$$2p^3 + r = 9pq$$

29.If α, β, and γ are the roots of the equation

$x^3 + px^2 + qx + r = 0$ **find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the coefficients.**

Soln:

$$\alpha + \beta + \gamma = -p; \qquad \alpha\beta\gamma = -r$$

$$\sum\frac{1}{\beta\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

30.If α and β are the roots of the quadratic equation $2x^2 - 7x + 13 = 0$ construct a quadratic equation whose roots are α^2 and β^2 . M-24

Soln: put $x^2 = y$ $\qquad 2y - 7\sqrt{y} + 13 = 0$

$$2y + 13 = 7\sqrt{y}$$

$$(2y + 13)^2 = \left(7\sqrt{y}\right)^2$$

$$4y^2 + 52y + 169 = 49y$$

$$4y^2 + 3y + 169 = 0$$

31.If α and β are the roots of the quadratic equation $17x^2 + 43x - 73 = 0$ construct a quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

Soln:

Roots increase by 2 then

equation diminish by -2

	17	43	-73
-2	0	-34	-18
	17	9	-91
	0	-34	
	17	-25	

Required equation is $17x^2 - 25x - 91 = 0$

32.Find the sum of squares of roots of

$$2x^4 - 8x^3 + 6x^2 - 3 = 0$$

Soln: Let α, β, γ and δ be the roots of

$$x^4 - \frac{8}{2}x^3 + \frac{6}{2}x^2 - \frac{3}{2} = 0$$

$$\alpha + \beta + \gamma + \delta = 4$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 3$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (4)^2 - 2(3)$$

$$= 10$$

PROBABILITY DISTRIBUTIONS

1.A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes

the total score in two throws. Find (i) The probability mass function. (ii) The cumulative distribution function.(iii) $P(3 \leq X < 6)$ (iv) $P(X \geq 4)$.

Soln: The random variable X takes the value 2,3,4,5 and 6.

X	2	3	4	5	6
PMF	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
CDF	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

+	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

(iii) $P(3 \leq X < 6) = P(x = 3) + P(x = 4) + P(x = 5)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$

(iv) $P(X \geq 4) = P(x = 4) + P(x = 5) + P(x = 6)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

2. A six sided die is marked '1' on one face, '3' on two of its faces ,and '5' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws. Find (i) The probability mass function (ii) The cumulative distribution function.(iii) $P(4 \leq X < 10)$ (iv) $P(X \geq 6)$.

Soln: The random variable X takes the value 2,4,6,8 and 10.

X	2	4	6	8	10
PMF	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
CDF	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

+	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

(iii) $P(4 \leq X < 10)$

$= P(x = 4) + P(x = 6) + P(x = 8)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$

(iv) $P(X \geq 6) = P(x = 6) + P(x = 8) + P(x = 10)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

3.A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2 < X < 6)$ (ii) $P(2 \leq X < 5)$ (iii) $P(X \leq 4)$ (iv) $P(3 < X)$

Soln: Given f is P.M.F

$\therefore \sum f(x) = 1$

$k + 2k + 6k + 5k + 6k + 10k = 1$

$30k = 1 \implies k = \frac{1}{30}$

x	1	2	3	4	5	6
$f(x)$	$\dfrac{1}{30}$	$\dfrac{2}{30}$	$\dfrac{6}{30}$	$\dfrac{5}{30}$	$\dfrac{6}{30}$	$\dfrac{10}{30}$

(i) $P(2 < X < 6) = P(x = 3) + P(x = 4) + P(x = 5) = \dfrac{6}{30} + \dfrac{5}{30} + \dfrac{6}{30} = \dfrac{17}{30}$

(ii) $P(2 \leq X < 5) = P(x = 2) + P(x = 3) + P(x = 4) = \dfrac{2}{30} + \dfrac{6}{30} + \dfrac{5}{30} = \dfrac{13}{30}$

(iii) $P(X \leq 4) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = \dfrac{1}{30} + \dfrac{2}{30} + \dfrac{6}{30} + \dfrac{5}{30} = \dfrac{14}{30}$

(iv) $P(3 < X) = P(x = 4) + P(x = 5) + P(x = 6) = \dfrac{5}{30} + \dfrac{6}{30} + \dfrac{10}{30} = \dfrac{21}{30}$

4. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) $P(2 \leq X < 5)$ (ii) $P(3 < X)$

Soln: Given f is P.M.F $\therefore \sum f(x) = 1$

$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$

$\qquad 6k^2 + 5k - 1 = 0$

$\qquad k = -1 \;,\; k = \dfrac{1}{6}$

x	1	2	3	4	5
$f(x)$	$\dfrac{1}{36}$	$\dfrac{2}{36}$	$\dfrac{3}{36}$	$\dfrac{2}{6} = \dfrac{12}{36}$	$\dfrac{3}{6} = \dfrac{18}{36}$

(i) $P(2 \leq X < 5) = P(x = 2) + P(x = 3) + P(x = 4) = \dfrac{2}{36} + \dfrac{3}{36} + \dfrac{12}{36} = \dfrac{17}{36}$

(ii) $P(3 < X) = P(x = 4) + P(x = 5) = \dfrac{12}{36} + \dfrac{18}{36} = \dfrac{30}{36}$

5.The cumulative distribution function of a discrete random variable is given by

find

      (i)   The Probability mass function $f(x)$

      (ii) $P(X < 3)$

      (iii)   (iii) $P(X \geq 2)$

$$F(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ \dfrac{1}{2} & ; 0 \leq x < 1 \\ \dfrac{3}{5} & ; 1 \leq x < 2 \\ \dfrac{4}{5} & ; 2 \leq x < 3 \\ \dfrac{9}{10} & ; 3 \leq x < 4 \\ 1 & ; 4 \leq x < \infty \end{cases}$$

Soln.

i)The values of the discrete random variable X

are 0,1,2,3,4.

x	0	1	2	3	4
$f(x)$	$\dfrac{1}{2} = \dfrac{5}{10}$	$\dfrac{1}{10}$	$\dfrac{2}{10}$	$\dfrac{1}{10}$	$\dfrac{1}{10}$

$ii) P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = \dfrac{5}{10} + \dfrac{1}{10} + \dfrac{2}{10} = \dfrac{8}{10} = \dfrac{4}{5}$

iii) $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) = \dfrac{2}{10} + \dfrac{1}{10} + \dfrac{1}{10} = \dfrac{4}{10} = \dfrac{2}{5}$

6.The cumulative distribution function of a discrete random variable is given by $F(x) = \begin{cases} 0 & ; -\infty < x < -1 \\ 0.15 & ; -1 \leq x < 0 \\ 0.35 & ; 0 \leq x < 1 \\ 0.60 & ; 1 \leq x < 2 \\ 0.85 & ; 2 \leq x < 3 \\ 1 & ; 3 \leq x < \infty \end{cases}$

Find (i)the probability mass function (ii)$p(X < 1)$ and(iii) $P(X \geq 2)$

The values of the discrete random variable X are $-1,0,1,2,3$.

Soln:

(i)The Probability mass function $f(x)$:

x	-1	0	1	2	3
$F(x)$	0.15	0.35	0.60	0.85	1
$f(x)$	0.15	0.20	0.25	0.25	0.15

$ii) P(X < 1) = P(X = -1) + P(X = 0) = 0.15 + 0.20 = 0.35$

iii) $P(X \geq 2) = P(X = 2) + P(X = 3) = 0.25 + 0.15 = 0.40$

INVERSE TRIGONOMETRY

5 Marks

Important Hints:

$tan^{-1}x + tan^{-1}y = tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \quad tan^{-1}x - tan^{-1}y = tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

$cos^{-1}x + cos^{-1}y = cos^{-1}(xy - \sqrt{(1-x^2)}\ \sqrt{1-y^2}\ )$

1) If $a_1, a_2, a_3 \dots a_n$ is an arithmetic progression with common difference d, prove that

$$tan\left[tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) + tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + \dots + tan^{-1}\left(\dfrac{d}{1+a_na_{n-1}}\right)\right] = \dfrac{a_n - a_1}{1+a_1a_n}$$

soln

$tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) = tan^{-1}\left(\dfrac{a_2-a_1}{1+a_1a_2}\right) = tan^{-1}a_2 - tan^{-1}a_1$

$tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) = tan^{-1}\left(\dfrac{a_3-a_2}{1+a_2a_3}\right) = tan^{-1}a_3 - tan^{-1}a_2$

$tan^{-1}\left(\dfrac{d}{1+a_na_{n-1}}\right) = tan^{-1}\left(\dfrac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right) = tan^{-1}a_n - tan^{-1}a_{n-1}$

$tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) + tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + \dots + tan^{-1}\left(\dfrac{d}{1+a_na_{n-1}}\right) = tan^{-1}a_n - tan^{-1}a_1$

$tan\left[tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) + tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + \dots + tan^{-1}\left(\dfrac{d}{1+a_na_{n-1}}\right)\right]$

$= tan[\ tan^{-1}a_n - tan^{-1}a_1]$

$= tan\left[tan^{-1}\left(\dfrac{a_n-a_1}{1+a_1a_n}\right)\right]$

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$$= \frac{a_n - a_1}{1 + a_1 a_n}$$

2) prove that: $tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), |x| < 1/\sqrt{3}$

soln

$$tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{x + \frac{2x}{1-x^2}}{1 - x\left(\frac{2x}{1-x^2}\right)}\right)$$

$$= tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right)$$

$$= tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

3) Show that $tan^{-1} x + tan^{-1} y + tan^{-1}z = tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Soln:

$$tan^{-1} x + tan^{-1} y + tan^{-1}z = tan^{-1}\left(\frac{x+y}{1-xy}\right) + tan^{-1}(z)$$

$$= tan^{-1}\left(\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right)$$

$$= tan^{-1}\left(\frac{[x+y+z(1-xy)]/(1-xy)}{[1-xy-(xz+yz)]/(1-xy)}\right)$$

$$= tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

4) IF $tan^{-1} x + tan^{-1} y + tan^{-1}z = \pi$ then show that $x + y + z = xyz$

Soln:

$$tan^{-1} x + tan^{-1} y + tan^{-1}z = tan^{-1}\left(\frac{x+y}{1-xy}\right) + tan^{-1}(z)$$

$$= tan^{-1}\left(\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right)$$

$$= tan^{-1}\left(\frac{[x+y+z(1-xy)]/(1-xy)}{[1-xy-(xz+yz)]/(1-xy)}\right)$$

$$tan^{-1} x + tan^{-1} y + tan^{-1}z = tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \pi$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = tan\pi = 0$$

$$x + y + z - xyz = 0$$

$$x + y + z = xyz$$

5) Find the number of solutions of the equation
$$tan^{-1}(x-1) + tan^{-1}(x) + tan^{-1}(x+1) = tan^{-1}(3x)$$

Soln: $tan^{-1}(x-1) + tan^{-1}(x+1) = tan^{-1}(3x) - tan^{-1}(x)$

$$tan^{-1}\left(\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right) = tan^{-1}\left(\frac{3x-x}{1+3x(x)}\right)$$

$$\frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2}$$

$$2x(1 + 3x^2) = 2x(2 - x^2)$$

$$2x + 6x^3 = 4x - 2x^3$$

∴ Given equation has 3 solutions

$$8x^3 - 2x = 0$$

6) Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.

Soln: Given $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$

$$\frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2) - (x-1)(x+1)} = \tan\frac{\pi}{4}$$

$$\frac{x^2 - x + 2x - 2 + x^2 + x - 2x - 2}{x^2 - 4 - (x^2 - 1)} = 1$$

$$\frac{2x^2 - 4}{x^2 - 4 - x^2 + 1} = 1$$

$$\frac{2x^2 - 4}{-3} = 1 \Rightarrow 2x^2 = -3 + 4$$

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

6) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $0 < x, y, z < 1$, then show that

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Soln

Let $\cos^{-1} x = \alpha, \cos^{-1} y = \beta$ ⟶ Then $x = \cos\alpha, y = \cos\beta$

Given, $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$$\cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$$

$$\cos^{-1}(xy - \sqrt{1 - x^2}\sqrt{1 - y^2}) = \cos^{-1}(-z)$$

$$-z = xy - \sqrt{1 - x^2}\sqrt{1 - y^2}$$

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

(handwritten)
$\cos^{-1}n + \cos^{-1}y + \cos^{-1}z = \pi$
$\cos^{-1}n + \cos^{-1}y + \cos^{-1}z = \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3}$
$\cos^{-1}x = \frac{\pi}{3}$ | $\cos^{-1}y = \frac{\pi}{3}$ | $\cos^{-1}z = \frac{\pi}{3}$
$x = \cos\frac{\pi}{3}$ | $y = \cos\frac{\pi}{3}$ | $z = \cos\frac{\pi}{3}$
$x = \frac{1}{2}$ | $y = \frac{1}{2}$ | $z = \frac{1}{2}$

Now,
$x^2 + y^2 + z^2 + 2xyz = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
$= 1$

7) Draw the curve $\sin x$ in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin^{-1}x$ in [-1,1]

$y = \sin x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \sin^{-1}x$
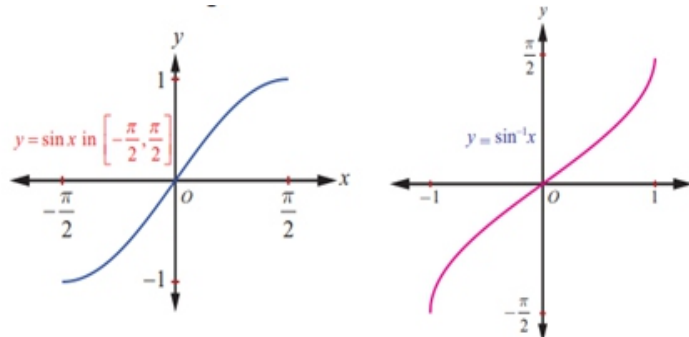
Domain : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ ⟶ Domain: $[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

8) Draw the curve $cos x$ in the domain $[0. \pi]$ and $cos^{-1}x$ in $[-1, 1]$
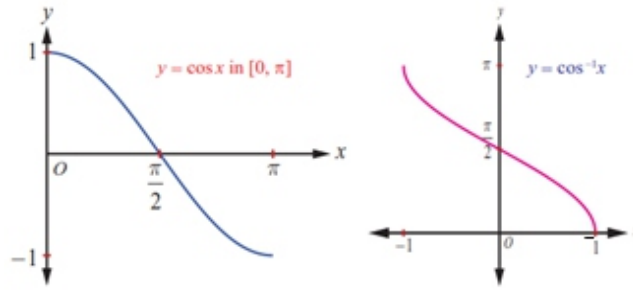
Domain:$[0, \pi] \rightarrow [-1, 1]$ Domain :$[-1, 1] \rightarrow [0, \pi]$

9) Draw the curve $tan\ x$ in the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $tan^{-1}x$ in R
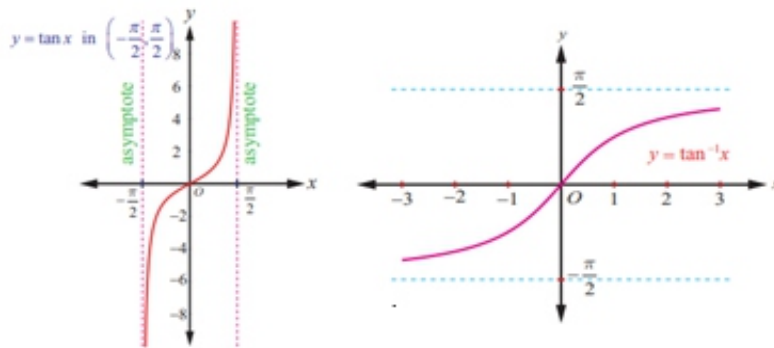
Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ Domain: $R \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

9.INTEGRAL CALCULUS

1. Find the area of the region bounded by the ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **J-22, M-24**

Soln: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

Area $A = 4 \int_0^a y dx$

$= 4 \int_0^a \frac{b}{a}\sqrt{a^2 - x^2} dx$

$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$
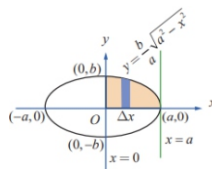
$= \frac{4b}{a} \times \frac{\pi a^2}{4}$

$= \pi ab$
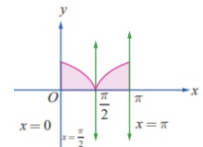
2. Find the area of the region bounded by $x -$ axis, the curve $y = |cos\ x|$ the lines $x = 0$ and $x = \pi$

Soln: $y = \begin{cases} cos x \ ; \ 0 \le x \le \frac{\pi}{2} \\ -cos x \ ; \ \frac{\pi}{2} \le x \le \pi \end{cases}$ **M-20**

$A = \int_0^\pi y dx$

$= 2 \int_0^{\frac{\pi}{2}} cos x dx$

$= 2(sin x)_0^{\frac{\pi}{2}}$

$= 2(1 + 0)$

$= 2$

3.Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$

Soln: Point of intersection $(0, 0) and\ (4, 4)$

$y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$

$A = \int_0^4 (y_1 - y_2) dx$

$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^3}{3 \times 4}\right]_0^4$
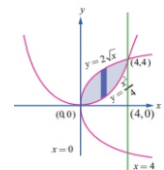
$$= \frac{4 \times 8}{3} - \frac{64}{12}$$

$$= \frac{16}{3}$$

4. Father of a family wishes to divide his square field bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

Soln: Point of intersection $(0, 0), (4, 4)$

$$A_1 = \int_0^4 y \, dx = \int_0^4 \frac{x^2}{4} \, dx$$

$$= \left(\frac{x^3}{12}\right)_0^4 = \frac{16}{3}$$

$$A_2 = \int_0^4 (y_1 - y_2) \, dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^3}{12}\right]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

$$A_3 = \int_0^4 x \, dy = \int_0^4 \frac{y^2}{4} \, dx = \left(\frac{y^3}{12}\right)_0^4 = \frac{16}{3}$$

5. Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y = |x|$.

Soln:

∴ the point of intersection $(0, 0)$, $(1, 1), (-1, 1)$

$$A = 2\int_0^1 (y_1 - y_2) \, dx$$

$$= 2\int_0^1 (x - x^2) \, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{3}$$
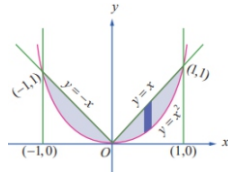
6. Find the area of the region bounded b/w the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

Soln:

$$A = \int_0^{\frac{\pi}{4}} (y_1 - y_2) \, dx + \int_{\frac{\pi}{4}}^{\pi} (y_1 - y_2) \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx - \int_{\frac{\pi}{4}}^{\pi} (\cos x - \sin x) \, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} - [\sin x + \cos x]_{\frac{\pi}{4}}^{\pi}$$

$$= 2\sqrt{2}$$

7. Find the area of the region bounded by $y = \cos x$, $y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Soln: $A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (y_1 - y_2) \, dx$
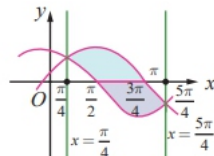
$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) \, dx$$

$$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= 2\sqrt{2}$$

8. Find, by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 = 0$ and the x-axis. J-23

Soln: Given $5x - 2y = 15 \Rightarrow x = \frac{15 + 2}{5} y$

$$x + y + 4 = 0 \Rightarrow x = -y - 4$$
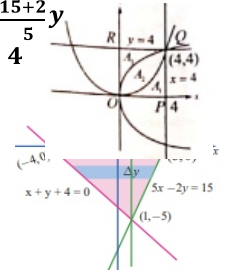
Point of intersection $(1, -5)$,

$$A = \int_{-5}^0 (x_1 - x_2) \, dy$$

$$= \int_{-5}^0 \left(\frac{15 + 2y}{5} - (-y - 4)\right) dy$$

$$= \int_{-5}^0 \left(7 + \frac{7y}{5}\right) dy$$

$$= \left(7y + \frac{7y^2}{10}\right)_{-5}^0$$

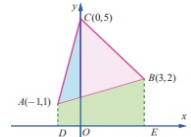$$= 0 - \left(-35 + \frac{35}{2}\right) = \frac{35}{2}$$

9. Using integration find the area of the region bounded by triangle ABC, whose vertices A, B, and C are $(-1,1)$, $(3, 2)$, and $(0,5)$ respectively.

Soln:

Equation AB $y = \frac{1}{4}(x + 5)$

Equation BC $y = -x + 5$

Equation AC $y = 4x + 5$

Area of △ ABC = Area of DACO + Area of OCBE −Area of DABE

$$= \int_{-1}^0 (4x + 5) \, dx + \int_0^3 (-x + 5) \, dx - \frac{1}{4}\int_{-1}^3 (x + 5) \, dx$$

$$= \left[4\frac{x^2}{2} + 5x\right]_{-1}^0 + \left[-\frac{x^2}{2} + 5x\right]_0^3 - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^3$$

$$= [0 - (2 - 5)] + \left[-\frac{9}{2} + 15 - 0\right] - \frac{1}{4}\left[\frac{9}{2} + 15 - \left(\frac{1}{2} - 5\right)\right]$$

$$= 3 + \frac{21}{2} - \frac{1}{4}\left(\frac{39}{2} + \frac{9}{2}\right)$$

$$= 3 + \frac{21}{2} - 6$$

$$= \frac{15}{2}$$

10. Using integration, find the area of the region which is bounded by x-axis, the tangent and normal to the circle $x^2 + y^2 = 4$ drawn at $(1, \sqrt{3})$.
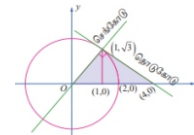
Soln:

$$A = \int_0^{\sqrt{3}} (x_1 - x_2) \, dy$$

$$= \int_0^{\sqrt{3}} \left[(4 - y\sqrt{3}) - \frac{y}{\sqrt{3}}\right] dy$$

$$= \left[4y - \sqrt{3}\frac{y^2}{2} - \frac{y^2}{2\sqrt{3}}\right]_0^{\sqrt{3}}$$
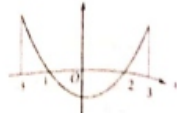
$$= 4\sqrt{3} - \frac{3}{2}\sqrt{3} - \frac{3}{2\sqrt{3}}$$

$$= 2\sqrt{3}$$

11. Find the area of the region bounded by the curve $2 + x - x^2 + y = 0$, x-axis, $x = -3$ and $x = 3$.

Soln: Given, $y = x^2 - x - 2$

$A = \int_{-3}^{-1}(y)dx + \int_{-1}^{2}(-y)dx + \int_{2}^{3}(y)dx$

$= \int_{-3}^{-1}(x^2 - x - 2)dx - \int_{-1}^{2}(x^2 - x - 2)dx + \int_{2}^{3}(x^2 - x - 2)dx$

$= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-3}^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-1}^{2} + \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{2}^{3}$
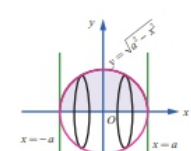
$= \frac{26}{3} + \frac{9}{2} + \frac{11}{6}$

$= 15$
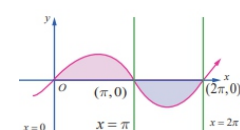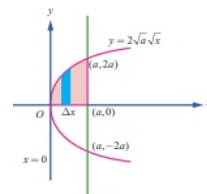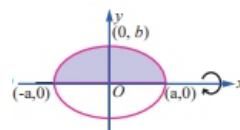
12. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Soln: Given $x^2 + y^2 = 16$, $y^2 = 6x$

$x^2 + 6x - 16 = 0$

$x = 2, \quad -8$ (not possible)

$A = 2\int_{0}^{2} ydx + 2\int_{2}^{4} ydx$

$= 2\int_{0}^{2}\sqrt{6x}\,dx + 2\int_{2}^{4}\sqrt{16 - x^2}dx$

$= 2\sqrt{6}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_{0}^{2} + 2\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}sin^{-1}\left(\frac{x}{4}\right)\right]_{2}^{4}$

$= \frac{16\sqrt{3}}{3} + 0 + 16 \times \frac{\pi}{2} - 4\sqrt{3} - 16 \times \frac{\pi}{6}$

$= \frac{4}{3}\left(4\pi + \sqrt{3}\right)$

1 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx$ **Soln:** Let $f(x) = x\cos x$ $\quad f(-x) = -f(x)$ $f(x)$ is an odd function $\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = 0$	**2** $\int_{-5}^{5}\frac{e^x - 1}{e^x + 1}dx$ **Soln:** Let $f(x) = \frac{e^x - 1}{e^x + 1}$ $f(-x) = -f(x)$ $f(x)$ is an odd function $\quad \int_{-5}^{5}\frac{e^x - 1}{e^x + 1}dx = 0$
3 $\int_{-5}^{5} x\cos\left(\frac{e^x - 1}{e^x + 1}\right)dx$ **Soln:** Let $f(x) = x\cos\left(\frac{e^x - 1}{e^x + 1}\right)$ $\quad f(-x) = -f(x) \Rightarrow f(x)$ is an odd function $\quad \int_{-5}^{5} x\cos\left(\frac{e^x - 1}{e^x + 1}\right)dx = 0$	**4** $\int_{-log2}^{log2} e^{-\lvert x\rvert}\,dx$ **Soln:** Let $f(x) = e^{-\lvert x\rvert} \Rightarrow f(x)$ is an even function $\int_{-log2}^{log2} e^{-\lvert x\rvert}\,dx = 2\int_{0}^{log2} e^{-x}\,dx$ $\qquad = 2(-e^{-x})_{0}^{log2}$ $\qquad = -2\left(e^{-log2} - e^{-0}\right)$ $\qquad = -2\left(\frac{1}{2} - 1\right)$ $\qquad = 1$
5 $\int_{0}^{1}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}dx$ **Soln:** $I = \int_{0}^{1}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}dx$..........(1) $\quad I = \int_{0}^{1}\frac{\sqrt{1-x}}{\sqrt{1-x}+\sqrt{x}}dx$.........(2) (1)+(2) $\Rightarrow$ $2I = \int_{0}^{1}\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{1-x}+\sqrt{x}}dx$ $\qquad = \int_{0}^{1}(1)dx$ $\quad 2I = (x)_{0}^{1} = 1$ $\quad \Rightarrow I = \frac{1}{2}$ $\int_{0}^{1}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}dx = \frac{1}{2}$	**6** $\int_{2}^{3}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$ **Soln:** $I = \int_{2}^{3}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$..........(1) $\quad I = \int_{2}^{3}\frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$.........(2) (1)+(2) $\Rightarrow$ $2I = \int_{2}^{3}\frac{\sqrt{5-x}+\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$ $\qquad = \int_{2}^{3}(1)\,dx$ $\qquad = (x)_{2}^{3}$ $\qquad = 3 - 2$ $\qquad = 1 \Rightarrow I = \frac{1}{2}$ $\int_{2}^{3}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx = \frac{1}{2}$

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7. $\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$ **M-22**

Soln: $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$ ....(1)

$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)} dx$ ....(2)

(1)+(2) $\Rightarrow$ $2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)} dx$

$2I = \int_0^a (1) dx$

$2I = (x)_0^a$

$\Rightarrow$ $2I = a - 0$ $\Rightarrow$ $I = \frac{a}{2}$

$\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx = \frac{a}{2}$

8. $\int_0^{\frac{\pi}{2}} \frac{f(sinx)}{f(sinx)+f(cosx)} dx$ **M-20**

Soln: $I = \int_0^{\frac{\pi}{2}} \frac{f(sinx)}{f(sinx)+f(cosx)} dx$ ....(1)

$I = \int_0^{\frac{\pi}{2}} \frac{f(cosx)}{f(cosx)+f(sinx)} dx$ ....(2)

(1)+(2) $\Rightarrow$ $2I = \int_0^{\frac{\pi}{2}} \frac{f(sinx)+f(cosx)}{f(sinx)+f(cosx)} dx$

$2I = \int_0^{\frac{\pi}{2}} (1) dx$

$2I = (x)_0^{\frac{\pi}{2}}$ $\Rightarrow$ $2I = \frac{\pi}{2} - 0$ $\Rightarrow$ $I = \frac{\pi}{4}$

$\int_0^{\frac{\pi}{2}} \frac{f(sinx)}{f(sinx)+f(cosx)} dx = \frac{\pi}{4}$

---

9. $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{tanx}} dx$ **M-24**

Soln:

$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\frac{sinx}{cosx}}} dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{cosx}}{\sqrt{cosx}+\sqrt{sinx}} dx$ ......(1)

$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{sinx}}{\sqrt{cosx}+\sqrt{sinx}} dx$ ....(2)

(1)+(2) $\Rightarrow$ $2I$ $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{cosx}+\sqrt{sinx}}{\sqrt{cosx}+\sqrt{sinx}} dx$

$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1) dx$

$2I = (x)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$ $\Rightarrow$ $2I = \frac{3\pi}{8} - \frac{\pi}{8}$

$\Rightarrow$ $2I = \frac{2\pi}{8}$ $\Rightarrow$ $I = \frac{\pi}{8}$

$\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{tanx}} dx = \frac{\pi}{8}$

10. $\int_0^1 |5x - 3| dx$

Soln:

$\int_{-4}^4 |x + 3| dx = \int_0^{\frac{3}{5}}(-5x + 3) dx + \int_{\frac{3}{5}}^1 (5x - 3) dx$

$= [-\frac{5x^2}{2} + 3x]_0^{\frac{3}{5}} + [\frac{5x^2}{2} - 3x]_{\frac{3}{5}}^1$

$= [(-\frac{9}{10} + \frac{9}{5}) - (0)] + [(\frac{5}{2} - 3) - (\frac{9}{10} - \frac{9}{5})]$

$= \frac{9}{5} - \frac{1}{2} + \frac{9}{5}$

$= \frac{9}{5} - \frac{1}{2}$

$= \frac{18-5}{10}$

$= \frac{13}{10}$

---

11. $\int_{-4}^4 |x + 3| dx$

Soln:

$\int_{-4}^4 |x + 3| dx = \int_{-4}^{-3}(-x - 3) dx + \int_{-3}^4 (x + 3) dx$

$= [-\frac{x^2}{2} - 3x]_{-4}^{-3} + [\frac{x^2}{2} + 3x]_{-3}^4$

$= [(-\frac{9}{2} + 9) - (-\frac{16}{2} + 12)] + [(\frac{16}{2} + 12) - (\frac{9}{2} - 9)]$

$= \frac{9}{2} - 4 + 20 + \frac{9}{2}$

$= 25$

12. (i) $\int xe^x dx$

Soln: $\int xe^x dx = xe^x - e^x + c$

$= e^x(x - 1) + c$

(ii) Show that $\int_0^1 xe^x dx = 1$

$\int_0^1 xe^x dx = [e^x(x - 1)]_0^1$

$= 0 - (-1) = 1$

---

13. $\int x^3 e^x dx$

Soln:

$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$

$= e^x(x^3 - 3x^2 + 6x - 6) + c$

14 $\int_0^{\frac{\pi}{2}} sin^{10}x dx$ **M-24**

$\int_0^{\frac{\pi}{2}} sin^{10}x dx = \frac{9 \times 7 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{63\pi}{512}$

---

15. $\int_0^{\frac{\pi}{2}} sin^9 x dx$

$\int_0^{\frac{\pi}{2}} sin^9 x dx = \frac{8 \times 6 \times 4 \times 2}{9 \times 7 \times 5 \times 3} = \frac{128}{315}$

16. $\int_0^{\frac{\pi}{2}} sin^6 x cos^4 x dx$

$\int_0^{\frac{\pi}{2}} sin^6 x cos^4 x dx = \frac{5 \times 3 \times 1 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{512}$

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17. $\int_0^{\frac{\pi}{2}} sin^5 x cos^4 x \, dx$ $\int_0^{\frac{\pi}{2}} sin^5 x cos^4 x \, dx = \frac{4 \times 2 \times 3 \times 1}{9 \times 7 \times 5 \times 3} = \frac{8}{315}$	**18.** $\int_0^{\frac{\pi}{2}} sin^3 x cos^5 x \, dx$ $\int_0^{\frac{\pi}{2}} sin^3 x cos^5 x \, dx = \frac{2 \times 4 \times 2}{8 \times 6 \times 4 \times 2} = \frac{1}{24}$
19. $\int_0^{2\pi} sin^7 \frac{x}{4} \, dx$ $\int_0^{2\pi} sin^7 \frac{x}{4} \, dx = 4 \int_0^{\frac{\pi}{2}} sin^7 x \, dx$ $\qquad = 4 \left[\frac{6 \times 4 \times 2}{7 \times 5 \times 3} \right] = \frac{64}{35}$	**20.** $\int_0^{\frac{\pi}{4}} sin^6 2x \, dx$ $\int_0^{\frac{\pi}{4}} sin^6 2x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} sin^6 x \, dx$ $\qquad = \frac{1}{2} \left[\frac{5 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2} \right] = \frac{5\pi}{64}$
21. $\int_0^1 x^3 (1-x)^4 dx$ $\int_0^1 x^3 (1-x)^4 dx = \frac{3! \times 4!}{(3+4+1)!} = \frac{3! \times 4!}{8!} = \frac{1}{280}$	**22.** $\int_0^1 x^5 \left(1-x^2\right)^5 dx$ $\int_0^1 x^5 \left(1-x^2\right)^5 dx = \frac{1}{2} \int_0^1 (x^2)^2 \left(1-x^2\right)^5 d(x^2)$ $\qquad = \frac{1}{2} \left[\frac{2! \times 5!}{8!} \right] = \frac{1}{336}$
23. $\int_0^{\infty} x^5 e^{-3x} dx$ **J-23** $\int_0^{\infty} x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$	**24.** $\int_b^{\infty} \frac{1}{a^2+x^2} dx$ **M-23** $\int_b^{\infty} \frac{1}{a^2+x^2} dx = \left[\frac{1}{a} tan^{-1} \frac{x}{a} \right]_b^{\infty}$ $\qquad = \left[\frac{1}{a} tan^{-1} \frac{\infty}{a} \right] - \left[\frac{1}{a} tan^{-1} \frac{b}{a} \right]$ $\qquad = \frac{1}{a} \left[\frac{\pi}{2} - tan^{-1} \frac{b}{a} \right]$
25. Find the area of the region bounded by $x-$ **axis, the sine curve** $y = sin\, x$ **, the lines** $x = 0$ **and** $x = 2\pi$ **.** **Soln:** $\quad A = \int_0^{2\pi} y \, dx$ $\qquad = 2 \int_0^{\pi} y \, dx$ $\qquad = 2 \int_0^{\pi} sin\, x \, dx$ $\qquad = 2(-cos\, x)_0^{\pi}$ $\qquad = 2(1+1)$ $\qquad = 4$	**26. Find the area of the region bounded by the curve** $y^2 = 4ax$ **and its latus rectum.** **J-22** **Soln: Given,** $y^2 = 4ax \Rightarrow y = 2\sqrt{a}\sqrt{x}$ **Area** $A = 2 \int_0^a y \, dx$ $\qquad = 2 \int_0^a 2\sqrt{a}\sqrt{x} \, dx$ $\qquad = 4\sqrt{a} \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right)_0^a$ $\qquad = \frac{8\sqrt{a} \times a^{\frac{3}{2}}}{3}$ $\qquad = \frac{8a^2}{3}$
27. Find the volume of the solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **,** $a > b$ **about the major axis.** **Soln:** $V = \pi \int_{-a}^a y^2 dx$ $\qquad = \pi \int_{-a}^a \frac{b^2}{a^2} (a^2 - x^2) dx$ $\qquad = 2\pi \frac{b^2}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a$ $\qquad = 2\pi \frac{b^2}{a^2} \left(a^3 - \frac{a^3}{3} \right)$ $\qquad = 2\pi \frac{b^2}{a^2} \left(\frac{2a^3}{3} \right)$ $\qquad = \frac{4\pi a b^2}{3}$	**28. Find the volume of sphere of radius a.** **Soln:** $V = \pi \int_{-a}^a y^2 dx$ $\qquad = \pi \int_{-a}^a (a^2 - x^2) dx$ $\qquad = 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$ $\qquad = 2\pi \left(a^3 - \frac{a^3}{3} \right)$ $\qquad = 2\pi \left(\frac{2a^3}{3} \right)$ $\qquad = \frac{4\pi a^3}{3}$

29. **Find the volume of right circular cone of base radius r and height h.**

Soln:
$$V = \pi \int_{-a}^{a} y^2 \, dx$$

$$= \pi \int_{0}^{h} \left(\frac{r}{h} x\right)^2 dx$$

$$= \pi \left(\frac{r}{h}\right)^2 \left[\frac{x^3}{3}\right]_0^h$$

$$= \pi \frac{r^2}{h^2} \left(\frac{h^3}{3}\right) \qquad y = \frac{rx}{h}$$

$$= \pi \left(\frac{r^2 h}{3}\right)$$

$$= \frac{1}{3} \pi r^2 h$$